Test #1 Solutions

1 Elimination (10 points)

Solve the following system of linear equations using elimination and back substitution. Circle the pivots as you discover them. Show your work!

\[
\begin{align*}
3x - 2y + 2z &= 10 \\
-6x + 2y + 4z &= -10 \\
3x + 2y - 3z &= 1 \\
\end{align*}
\]

Solution: The first pivot is the 3 in the first equation.

\[
\begin{align*}
3x - 2y + 2z &= 10 \\
\text{(Row 2)} + (2 \cdot \text{Row 1}) &\quad -2y + 8z = 10 \\
3x + 2y - 3z &= 1 \\
\end{align*}
\]

The second pivot is the \(-2\) in the second equation.

\[
\begin{align*}
3x - 2y + 2z &= 10 \\
-2y + 8z &= 10 \\
\text{(Row 3) - (Row 1)} &\quad 4y - 5z = -9 \\
\end{align*}
\]

\[
\begin{align*}
3x - 2y + 2z &= 10 \\
-2y + 8z &= 10 \\
\text{(Row 3) + (2 \cdot \text{Row 2})} &\quad 11z = 11 \\
\end{align*}
\]

The third pivot is the 11 in the third equation. Now we use back substitution: solving the third equation gives \(z = 1\). Then the second equation becomes \(-2y + 8 = 10\), so \(y = -1\). Substituting into the first equation yields \(3x - 2(-1) + 2 = 10\), so \(x = 2\).

Comments: When you’re trying to eliminate a term in a row using a pivot above it (like eliminating the \(-6x\) using the \(3x\) in the first step above), always add or subtract a multiple of the pivot row from that row. Don’t scale the row you’re working on because that will mess up the pivots!
2 Trouble with elimination (10 points)

Write down a system of two linear equations in two unknowns $x$ and $y$ such that:

(a) The system of equations has a unique solution, but elimination requires a row swap. (4 points)

*Possible solution:*

\[
\begin{align*}
y &= 2 \\
x - y &= 1
\end{align*}
\]

*Comments:* How do row swaps come up in elimination? When one of the pivots would be 0 (which is not allowed!), look at the rows below that pivot. If there is a nonzero coefficient below the 0, swap rows so that the nonzero coefficient becomes your pivot, and then proceed with elimination. If there is no nonzero coefficient below the 0, then row swaps won’t fix the problem, so elimination breaks down and the system of equations has either no solution or infinitely many solutions.

(b) The system of equations has no solutions. (3 points)

*Possible solution:*

\[
\begin{align*}
x + y &= 2 \\
x + y &= 3
\end{align*}
\]

*Comments:* The geometric picture is two parallel lines, which don’t intersect!

(c) The system of equations has infinitely many solutions. (3 points)

*Possible solution:*

\[
\begin{align*}
x - y &= 2 \\
3x - 3y &= 6
\end{align*}
\]

*Comments:* Both equations give the same line, so any point on that line is a solution to the system of equations!
3 Inventions (12 points)

(a) Invent vectors $\vec{v}$ and $\vec{w}$ in three dimensions such that the set of all linear combinations of $\vec{v}$ and $\vec{w}$ is a plane. (2 points)

Possible solution:

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Comments: $\vec{v}$ and $\vec{w}$ should lie on different lines through the origin (the span of $\vec{v}$ should not contain $\vec{w}$). This is true for most pairs of vectors in three dimensions.

(b) Invent vectors $\vec{v}$ and $\vec{w}$ in three dimensions such that the set of all linear combinations of $\vec{v}$ and $\vec{w}$ is a line. (2 points)

Possible solution:

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Comments: $\vec{v}$ and $\vec{w}$ should lie on the same line through the origin (the span of $\vec{v}$ is the same as the span of $\vec{w}$).

(c) Invent a unit vector $\vec{u}$ in four dimensions with no component equal to zero. (2 points)

Possible solution:

$$\vec{u} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Comments: You can make any nonzero vector into a unit vector by dividing it by its length! That’s how I arrived at the above solution.
(d) Invent two perpendicular unit vectors \( \vec{u} \) and \( \vec{U} \) in five dimensions. (3 points)

*Possible solution:*

\[
\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{U} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

*Comments:*

Read the question carefully! Many of you did not notice that the problem demands that \( \vec{u} \) and \( \vec{U} \) be unit vectors! The secret reason why I wanted your perpendicular vectors to have length 1 was to prevent you from using \( \vec{u} = \vec{0} \) or \( \vec{U} = \vec{0} \).

(e) Invent a 3 \( \times \) 3 matrix \( P \) such that \( P \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ x \\ y \end{bmatrix} \). (3 points)

*Solution:*

\[
P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\]
4 Interpreting $A \vec{x} = \vec{b}$ (10 points)

Consider the matrix equation
\[
\begin{bmatrix}
-1 & 2 \\
1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
\end{bmatrix} =
\begin{bmatrix}
-4 \\
1 \\
\end{bmatrix}.
\]

(a) Row picture: write down the system of equations associated to the matrix equation. (2 points)

Solution:
\[
-x + 2y = -4 \\
x + y = 1
\]

(b) Solve the system of equations in (a) geometrically by drawing a picture. (6 points)

Description of solution: The solution set of each equation in (a) is a line in the plane. You should sketch these two lines and note that they intersect at the vector \[ \begin{bmatrix} 2 \\ -1 \end{bmatrix}. \]

(c) Column picture: write down the linear combination associated to the solution you found in (b). (2 points)

Solution:
\[
2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}
\]
5 Dot products (8 points)

(a) Using the dot product, write down the definition of the length $\|\vec{v}\|$ of a vector $\vec{v}$. (2 points)

Solution:

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$

(b) Using the dot product, write down the definition of two vectors $\vec{v}$ and $\vec{w}$ being perpendicular. (2 points)

Solution:

$$\vec{v} \cdot \vec{w} = 0$$

(c) Suppose $\vec{v}$ and $\vec{w}$ are perpendicular. Show that $\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$. (This is the Pythagorean theorem!

Hint: expand the left side using your definition in (a) on the vector $\vec{v} - \vec{w}$, do some algebra, and use the fact that $\vec{v}$ and $\vec{w}$ are perpendicular. (4 points)

Solution:

$$\|\vec{v} - \vec{w}\|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w})$$

by (a)

$$= \vec{v} \cdot \vec{v} - 2\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w}$$

$$= \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w}$$

since $\vec{v} \cdot \vec{w} = 0$

$$= \|\vec{v}\|^2 + \|\vec{w}\|^2$$

by (a)

Comments: When writing a proof that an equation is true, you should never start by writing that equation and then manipulating it. Many of you started with the equation $\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$, which is what you are trying to show, and manipulated the left side until you got the equation $\|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$, which is clearly true. Demonstrating that the equation you want to prove can be used to deduce a clearly true equation is a logically flawed method of proof!

For example, begin with the equation $1 = 0$. Multiplying both sides by 0 gives the equation $0 = 0$, which is clearly true. Does this prove that $1 = 0$? Of course not!

The usual way to prove an equation is true is to start with just one side of the equation and manipulate it until you get to the other side of the equation. This is what is done in the solution above! It is typically easier to start with the more complicated half of the equation (above, this was the left side) and simplify it until you get the other side of the equation.

Think of a proof as a deduction process. If you are trying to deduce that $\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$ is true, then it makes no sense to begin by asserting that $\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$ is true. Instead, arriving at that equation is the grand conclusion of the proof!