Math 2270 Final Exam Study Guide

December 7, 2014

1 Overview

1.1 Logistics

- Time and place: Tuesday, December 16, 1-3 PM, in ST 208 (our usual MWF classroom, which is not the usual Tuesday room!). We will start right on time, so come a few minutes early!

- Roughly twice as many problems as a test, but 20 minutes longer than a double test (120 = 2 \cdot 50 + 20).

- Worth 20% of the final grade.

- No “cheat sheets” or calculators.

1.2 Possible types of questions

- Inventions

- Computations

- Short answers

- Drawings to show geometry

- Simple proofs (as in past tests)

- True or false (possible but unlikely)

1.3 Studying advice

- The best way to prepare for the final exam is to carefully study all previous quizzes and tests. Ask me or a fellow student if you don’t understand a problem on a quiz or test. Focus especially on the tests. The majority of the final exam will be similar to the five tests.
• Material from Chapters 3, 6, and 7 is the most important because it incorporates the majority of the material in the other chapters. But any material we covered in the course could show up on the exam.

• Try not to leave anything blank! If you’re stumped on an invention, take a guess! If you’re having trouble on the first part of a multiple-part problem, guess a reasonable answer so that you can get partial credit on the later parts.

• Expect both computational and conceptual questions.

• Come to the review sessions on Thursday and Friday! If you want to meet me at another time, e-mail me to set up an appointment.

• The outline below summarizes some of the most important material. I do not promise that it is exhaustive!

2 Outline of Key Material

2.1 Some key definitions you should look up

• Inverse of a matrix. To check if a matrix $B$ is the inverse of $A$, compute $BA$ or $AB$ and see whether you get the identity $I$.

• Subspace. Think of examples and non-examples. To show that a set is a subspace, you have to show the two defining properties hold for all scalars $c$ and all vectors $\vec{v}, \vec{w}$ in the subspace. To show that a set is not a subspace, you only need to show that one of the properties fails for a particular choice of $c$ and $\vec{v}$ or $\vec{v}$ and $\vec{w}$.

• Independent vectors. One way to show vectors are independent is to construct a matrix whose columns are those vectors and show that the nullspace is $\{0\}$. If the nullspace is larger, then the vectors are dependent.

• Span of vectors. To show a vector $\vec{b}$ is in the span of some vectors, find a combination of those vectors that produces $\vec{b}$. If you can’t guess a combination, construct a matrix $A$ whose columns are those vectors and show that $A\vec{x} = \vec{b}$ has a solution. If $A\vec{x} = \vec{b}$ has no solution, then $\vec{b}$ is not in the span of the columns of $A$.

• Basis of a subspace. To show that vectors form a basis, construct a matrix whose columns are those vectors and show that the matrix is invertible (for instance by showing its determinant is nonzero). If the matrix is not invertible, then its columns are not a basis.

• Dimension of a subspace.

• Four fundamental subspaces $N(A)$, $C(A)$, $N(A^T)$, $C(A^T)$.

• Defining properties of the determinant.
• Eigenvalues and eigenvectors. To show that \( \bar{x} \) is an eigenvector of \( A \) with eigenvalue \( \lambda \), compute \( A\bar{x} \) and check whether you get \( \lambda\bar{x} \). If \( A\bar{x} \) does not equal a non-zero scalar times \( \bar{x} \), then \( \bar{x} \) is not an eigenvector of \( A \). Why is the definition of eigenvalues equivalent to the description of eigenvalues as roots of the polynomial \( \det(A - \lambda I) = 0 \)?

• Similar matrices. What properties are the same or not the same?

• Linear transformation. Think of examples and non-examples and how to show a function is or isn’t linear.

2.2 Solving \( A\bar{x} = \bar{0} \)

• Same problem as computing the nullspace \( N(A) \).

• \( \bar{x} = \bar{0} \) is always a solution. Are there more solutions?

• Row picture of solutions (usual way to multiply \( A \) and \( \bar{x} \)): intersection of hyperplanes through the origin; orthogonal complement of the row space \( C(A^T) \).

• Column picture of solutions (new way to multiply \( A \) and \( \bar{x} \)): linear combinations of the columns of \( A \) that produce \( \bar{0} \).

• Solving: use elimination to turn \( A \) into \( U \). Back substitution yields the special solutions that are a basis for \( N(A) \) (the number of special solutions is \( \dim N(A) \)).

• When \( A \) is square and has enough pivots, the elimination process can also be viewed as finding the matrix factorization \( PA = LU \) (just \( A = LU \) if there are no row swaps). In this case \( A \) is invertible, so the only solution is \( \bar{x} = \bar{0} \).

2.3 Solving \( A\bar{x} = \bar{b} \)

• When \( \bar{b} \neq \bar{0} \), the set of solutions is not a subspace. Instead, it is a translation of the nullspace \( N(A) \) by any particular solution \( \bar{x}_p \).

• There could be no solution (when \( \bar{b} \) is not in the column space \( C(A) \)), one solution (when \( \bar{b} \) is in \( C(A) \) and \( N(A) = \{\bar{0}\} \)), or infinitely many solutions (when \( \bar{b} \) is in \( C(A) \) and \( \dim N(A) \geq 1 \)).

• Row picture of solutions: intersection of hyperplanes that may not go through the origin.

• Column picture of solutions: linear combinations of the columns of \( A \) that produce \( \bar{b} \).

• Solving: use elimination on the augmented matrix \( [A \bar{b}] \), find a particular solution \( \bar{x}_p \) (for instance by setting all the free variables to \( 0 \) and using back substitution), and add all linear combinations of the special solutions of \( A\bar{x} = \bar{0} \) (ignore \( \bar{b} \) when finding special solutions).

• In case \( A \) is invertible, the unique solution is \( \bar{x} = A^{-1}\bar{b} \).
2.4 Four fundamental subspaces

- Let $A$ be $m \times n$ of rank $r$.
- Basis for the nullspace $N(A)$: the special solutions to $A\tilde{x} = \tilde{0}$.
- Basis for the column space $C(A)$: the free columns of $A$ (use elimination to decide which columns are free, but use the corresponding columns of $A$, not of $U$, for the basis).
- Basis for the row space $C(A^T)$: the free rows of $A$ (or free columns of $A^T$).
- Basis for the left nullspace $N(A^T)$: the special solutions to $\tilde{x}^T A = \tilde{0}$ or $A^T \tilde{x} = \tilde{0}$.
- Dimensions: $\dim N(A) = n - r$, $\dim C(A) = r$, $\dim C(A^T) = r$, $\dim N(A^T) = m - r$.
- Orthogonality: $N(A)$ and $C(A^T)$ are orthogonal complements in $\mathbb{R}^n$; $C(A)$ and $N(A^T)$ are orthogonal complements in $\mathbb{R}^m$. This should be clear from the definitions!

2.5 Invertible matrices

- Let $A$ be a square $n \times n$ matrix. Then $A$ is invertible if and only if any one of the following holds: $r = n$; $N(A) = \{\tilde{0}\}$; $\det A = 0$; $0$ is not an eigenvalue of $A$.
- If $A$ is invertible, then $A\tilde{x} = \tilde{b}$ has a unique solution $\tilde{x} = A^{-1}\tilde{b}$ for every $\tilde{b}$.
- Computing $A^{-1}$: use Gauss-Jordan elimination on the augmented matrix $[A \ I]$, or use the cofactor formula $A^{-1} = \frac{1}{\det A} C^T$.
- It is worth remembering the general formula for the inverse of a $2 \times 2$ matrix (which comes from the cofactor formula!).

2.6 Eigenvalues and eigenvectors

- $A$ a square $n \times n$ matrix.
- Computing eigenvalues: roots of the polynomial $\det(A - \lambda I)$. The number of times each root $\lambda$ occurs is the multiplicity $m_\lambda$ of that eigenvalue.
- Computing independent eigenvectors: compute the special solutions of the nullspace $N(A - \lambda I)$ for each of the eigenvalues $\lambda$. The number of independent eigenvectors for the eigenvalue $\lambda$ is between $1$ and $m_\lambda$.
- If $A$ has $n$ independent eigenvectors, then $A$ is diagonalizable: $A = S\Lambda S^{-1}$, where $S$ is the eigenvector matrix and $\Lambda$ is the eigenvalue matrix.
- If $A$ is symmetric then it has only real eigenvalues and $n$ independent eigenvectors that can be chosen orthogonal. Then $A$ is diagonalizable: $A = Q\Lambda Q^T$ with $Q$ orthogonal.
• If all eigenvalues are $> 0$, then $A$ is positive definite. This can be checked by showing that all upper left determinants are $> 0$.

• Jordan form. The eigenvalues are on the diagonal, and there is one eigenvector for each Jordan block.

• Singular value decomposition (SVD) $A = U\Sigma V^T$ and the pseudoinverse.

2.7 Linear transformations

• Algebraic definition of linear transformation: $V \xrightarrow{T} W$ satisfies $T(cv) = cT(v)$ and $T(v + w) = T(v) + T(w)$.

• Geometric description: takes lines to lines and parallelograms to parallelograms (possibly collapsed).

• Visualizing transformations in the plane.

• Turning $V$ into $\mathbb{R}^n$: choose a basis $B$ for $V$ to write vectors in $V$ using $B$-coordinates.

• Thinking of $T$ as multiplication by a matrix: choose bases $B$ for $V$ and $B'$ for $W$ and see how $T$ acts on the basis vectors.

• Change of basis matrices: matrices for the identity transformation obtained by choosing two different bases (usually one basis will be the standard basis, in which case the columns of the change of basis matrix are exactly the vectors in the other basis).

2.8 Applications and miscellaneous

• Solving a word problem describing a system of equations (as in Quiz #2 Problem 2).

• Using Gram-Schmidt to turn a basis into an orthonormal basis.

• Computing a best-fit line or parabola using projection (Section 4.3).

• Computing a determinant by cofactor expansion.

• Computing powers of a matrix using diagonalization.

• Solving differential equations in several variables using eigenvectors or the matrix exponential $e^{At}$.

• Computing the SVD.