Learning Celebration #1 Solutions

Example (10 points)

Give an example of each of the following or state that no example exists:

(1) A bounded sequence that doesn’t have a limit. (2 points)
   \(-1, 1, -1, \ldots, (-1)^n, \ldots\)

(2) A subset of \(\mathbb{R}\) that has a supremum but does not have a least upper bound. (2 points)
   \(\mathbb{Z}\)

(3) A sequence that converges to \(\sqrt{3}\) and has maximum 1000. (2 points)
   \(1000, \sqrt{3}, \sqrt{3}, \sqrt{3}, \ldots\)

(4) A subset of \(\mathbb{R}\) that has no infimum. (2 points)
   No example exists. (You also get full points for “the empty set”.)

(5) An ordered field that is not complete. (2 points)
   \(\mathbb{Q}\)

Computation (5 points)

(6) Let \((0, \infty) \rightarrow \mathbb{R}\) be the function with rule \(f(x) = \frac{1}{x}\). Compute \(\text{inf}_{(0,1)} f\). (3 points)
   Draw the graph of the function to see that \(f(x)\) goes to \(\infty\) as \(x \to 0\) from the right, and
   that \(f(x)\) decreases to 1 as \(x \to 1\) from the left. Thus \(\text{inf}_{(0,1)} f = 1\).

(7) Compute the first five terms of the sequence \(\left\{ \frac{n^2+3}{\sqrt{n}} \right\} \). (2 points)
   \(4, \frac{7}{\sqrt{2}}, \frac{12}{\sqrt{3}}, \frac{19}{\sqrt{2}}, \frac{28}{\sqrt{3}}\)

Precision (5 points)

(8) Define what it means for a sequence \(\{a_n\}\) of real numbers to converge to \(a \in \mathbb{R}\). (3 points)
   A sequence \(\{a_n\}\) of real numbers converges to \(a \in \mathbb{R}\) if for all \(\epsilon > 0\), there is \(N \in \mathbb{R}\) such
   that for all \(n > N\), \(|a_n - a| < \epsilon\).

(9) Define what it means for \(\mathbb{R}\) to have the Archimedean property. (2 points)
   \(\mathbb{R}\) has the Archimedean property, which means that for all \(x \in \mathbb{R}\) there is \(n \in \mathbb{N}\) such
   that \(n > x\).
Proof (20 points)

(10) Guess the limit of the sequence \( \left\{ \frac{1}{n+1} \right\} \), write your guess as a claim, and prove that the claim is correct. (5 points)

Claim. \( \lim_{n \to \infty} \frac{1}{n+1} = 0. \)

Proof. Given \( \epsilon > 0 \), let \( N = \frac{1}{\epsilon} \). Then for all \( n > N \),
\[
\left| \frac{1}{n+1} - 0 \right| = \frac{1}{n+1} < \frac{1}{n} < \frac{1}{N} = \epsilon.
\]

\( \square \)

(11) Guess the limit of the sequence \( \left\{ \frac{2n}{3n-1} \right\} \), write your guess as a claim, and prove that the claim is correct. (5 points)

Claim. \( \lim_{n \to \infty} \frac{2n}{3n-1} = \frac{2}{3}. \)

Proof. Given \( \epsilon > 0 \), let \( N = \frac{1}{3\epsilon} \). Then for all \( n > N \),
\[
\left| \frac{2n}{3n-1} - \frac{2}{3} \right| = \frac{2}{9n-3} \leq \frac{2}{6n} = \frac{1}{3n} < \frac{1}{3N} = \epsilon.
\]

\( \square \)

(12) Guess the limit of the sequence \( \left\{ \sqrt{n^2 - 1} - n \right\} \), write your guess as a claim, and prove that the claim is correct. (5 points)

Claim. \( \lim_{n \to \infty} \sqrt{n^2 - 1} - n = 0. \)

Proof. Given \( \epsilon > 0 \), let \( N = \frac{1}{\epsilon} \). Then for all \( n > N \),
\[
|\sqrt{n^2 - 1} - n| = \left| \frac{-1}{\sqrt{n^2 - 1} + n} \right| = \frac{1}{\sqrt{n^2 - 1} + n} \leq \frac{1}{n} < \frac{1}{N} = \epsilon.
\]

\( \square \)

(13) Prove that if \( \inf A = -\infty \), then for each \( n \in \mathbb{Z} \) there is an element \( a_n \in A \) such that \( a_n < n. \) (5 points)

Proof. Let \( n \in \mathbb{Z} \) be given. Since \( \inf A = -\infty \), \( n \) is not a lower bound for \( A \), so there is \( a_n \in A \) such that \( a_n < n. \)

\( \square \)