MATH 2200 Quiz #3 Solutions

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Comments on proof by induction:

- If you set up a proof by induction correctly, then you got most of the points. For Theorem 2, many of you had trouble getting to $(k + 2)! - 1$ while trying to show that the theorem is true for $n = k + 1$, but as long as you did everything else correctly, your efforts earned you 8 out of 10 points.

- Big problems in a proof by induction are not checking the base case, not clearly stating what your assumption is (“suppose the theorem is true for some $n = k \in \mathbb{Z}^{\geq 1}$”), and doing strange things in an effort to deduce that the theorem must be true for $k + 1$.

- Please do not intentionally write false things in a proof! If your proof isn’t quite working out, don’t just jump to the conclusion you want. For instance, some of you proved Theorem 1 (below) like this:

> **Proof of Theorem 1.** Induction on $n$. For the base case $n = 1$, note that the sum of the first 1 positive integer is 1, which equals $1^2$. Now suppose the theorem is true for some $n = k \in \mathbb{Z}^{\geq 1}$, namely $1 + 3 + \cdots + (2k + 1) = k^2$. Adding the next odd positive integer $(2k + 3)$ to both sides, we get

$$1 + 3 + \cdots + (2k + 1) + (2k + 3) = k^2 + (2k + 3) = (k + 1)^2,$$

which proves the theorem for $n = k + 1$. Thus we are done by induction.

The main problem is that $k^2 + (2k + 3) = (k + 1)^2$ is completely false! It would have been much better to get to $k^2 + (2k + 3)$, realize that this is not equal to $(k + 1)^2$, and write a small note that you want to get $(k + 1)^2$, but you must have made a mistake because it is not coming out right. Of course, it would be even better if you realized that you’re not getting the right answer because $1 + 3 + \cdots + (2k + 1)$ is not the sum of the first $k$ odd positive integers: the sum should only be up to $2k - 1$.

- Finally, I think I made a big mistake in teaching you to prove the inductive step by “adding something to both sides” of the equation you get by assuming the theorem is true for some $n = k$. Some of you want to add $k + 1$ to both sides no matter what the context is, which often makes no sense. Here’s a slightly different way of thinking about the inductive step. Do these three steps on your scratch paper:

  (a) Write down what the theorem says in the case $n = k$. (This is what you are assuming to be true.)
  (b) Write down what the theorem says in the case $n = k + 1$. (This is what you are trying to prove!)
  (c) Try to prove (b) using the assumption (a).

With this approach, it should be completely clear what you are trying to show. There’s no mysterious term being added to both sides of an equation. For instance, suppose you’re trying to prove Theorem 1 below. On your scratch paper, write

(a) Assumption: $1 + 3 + \cdots + (2k - 1) = k^2$.
(b) Goal: $1 + 3 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2$. 

In order to prove an equation like the goal, you should start with one side of the goal equation and keep working on it until you get the other side. Naturally, you should try to use your assumption at some point. The key is to notice that the first $k$ terms in the left side of the goal equation are the same as the terms in the left side of the assumption equation. Thus you can use the assumption to deduce that:

$$1 + 3 + \cdots + (2k - 1) + (2k - 3) = \left[1 + 3 + \cdots + (2k - 1)\right] + (2k + 1) = k^2 + (2k + 1).$$

Then all that remains is to note that $k^2 + (2k + 1)$ can be factored as $(k + 1)^2$, which gives you the right side of the goal equation. Thus you have successfully proved the goal equation using the assumption!

- I've written solutions to Theorems 1 and 2 (below) in this style, without any mysterious adding of terms to both sides of the assumption equation. Please read through my proofs and make sure they make sense in the context of steps (a), (b), and (c) above.

- If you are still having trouble with proof by induction, please come see me!!

1 Proof by Induction (20 points)

Prove the following two theorems by induction. Write complete sentences! (10 points each)

**Theorem 1.** Let $n \in \mathbb{Z}^{>1}$. Then the sum of the first $n$ odd positive integers is $n^2$.

**Solution:**

Proof of Theorem 1. Induction on $n$. For the base case $n = 1$, note that the sum of the first 1 positive integer is 1, which equals $1^2$. Now suppose the theorem is true for some $n = k \in \mathbb{Z}^{>1}$, namely $1 + 3 + \cdots + (2k - 1) = k^2$. Then we compute

$$1 + 3 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2,$$

which proves the theorem for $n = k + 1$. Thus we are done by induction. \( \square \)

**Theorem 2.** Let $n \in \mathbb{Z}^{>1}$. Then $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$.

**Solution:**

Proof of Theorem 2. Induction on $n$. For the base case $n = 1$, note that $1 \cdot 1! = 1 = (1 + 1)! - 1$. Now suppose the theorem is true for same $n = k \in \mathbb{Z}^{>1}$, namely that $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k + 1)! - 1$. Then we compute

$$1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k + 1) \cdot (k + 1)! = (k + 1)! - 1 + (k + 1) \cdot (k + 1)!$$

$$= (1 + k + 1) \cdot (k + 1)! - 1$$

$$= (k + 2)! - 1,$$

which proves the theorem for $n = k + 1$. Thus we are done by induction. \( \square \)

2 Permutations and Combinations (10 points)

Let $S = \{a, b, c, d, e\}$ be a set with 5 elements.

(a) Define “$r$-permutation of $S$”. (2 points)

**Solution:** An $r$-permutation of $S$ is a sequence of $r$ distinct elements of $S$.

(b) Give an example of a 3-permutation of $S$. (2 points)

**Solution:** $(b, c, e)$. 

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(c) Define “r-combination of S”. (2 points)
Solution: An r-combination of S is a subset of S containing r elements.

(d) Give an example of a 3-combination of S. (2 points)
Solution: \{a, b, d\}.

(e) How many 3-combinations of S are there? (2 points)
Solution: Since S has 5 elements, the number of 3-combinations of S is
\[ \binom{5}{3} = \frac{5!}{(5-3)!3!} = 10. \]

3 Binomial Theorem (10 points)

(a) State the binomial theorem. (3 points)
Solution: Let \( n \in \mathbb{Z}^{\geq 1} \) and \( x, y \) be variables. Then
\[ (x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n. \]

(b) Use Pascal’s triangle to compute the binomial coefficients \( \binom{5}{r} \) for each integer \( 0 \leq r \leq 5 \). (3 points)
Solution: The first six rows of Pascal’s triangle are:

\[
\begin{array}{ccccccc}
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 2 & 1 & & & & \\
1 & 3 & 3 & 1 & & & \\
1 & 4 & 6 & 4 & 1 & & \\
1 & 5 & 10 & 10 & 5 & 1 & \\
\end{array}
\]

Thus \( \binom{5}{0} = 1, \binom{5}{1} = 5, \binom{5}{2} = 10, \binom{5}{3} = 5, \binom{5}{4} = 1, \) and \( \binom{5}{5} = 1. \)

(c) Use the binomial theorem and (b) to expand \((2z - 1)^5\). (4 points)
Solution: Taking \( x = 2z \) and \( y = -1 \) in the binomial theorem, we get
\[
(2z - 1)^5 = \binom{5}{0}(2z)^5 + \binom{5}{1}(2z)^4(-1) + \binom{5}{2}(2z)^3(-1)^2 + \binom{5}{3}(2z)^2(-1)^3 + \binom{5}{4}(2z)(-1)^4 + \binom{5}{5}(-1)^5
= 32z^5 - 5 \cdot 2^4 z^4 + 10 \cdot 2^3 z^3 - 10 \cdot 2^2 z^2 + 5 \cdot 2z - 1
= 32z^5 - 80z^4 + 80z^3 - 40z^2 + 10z - 1.
\]