Math 2200 Homework 9 Proofs

Degree and Simple Graphs

Exercise 1. Prove the following theorem:

**Theorem.** Let $G = (V, E, \phi)$ be a graph. Then the sum of the degrees of all vertices in $G$ is equal to twice the number of edges of $G$, namely

$$
\sum_{v \in V} \deg(v) = 2 \cdot |E|.
$$

**Proof.** Let $e \in E$ be any edge. If $e$ is a loop, then $e$ contributes 2 to the degree of its only endpoint. If $e$ is not a loop, then $e$ contributes 1 to the degree of each of its two endpoints. In either case, $e$ contributes 2 to the sum of the degrees of all the vertices. Since this is true for every edge $e$, we see that

$$
\sum_{v \in V} \deg(v) = 2 \cdot |E|.
$$

\[\square\]

Exercise 2. Prove the following theorem:

**Theorem.** Let $G = (V, E, \phi)$ be a simple graph and let $n = |V|$ be the number of vertices of $G$. Then $|E|$, the number of edges of $G$, satisfies the inequality

$$
|E| \leq \binom{n}{2}.
$$

**Proof.** Since $G$ is simple, $G$ contains no loops or multiple edges. Thus every edge of $G$ has two distinct endpoints, and there is at most one edge between every pair of distinct vertices. So the most edges $G$ can have is one for every pair of distinct vertices in $G$. Since there are $\binom{n}{2}$ pairs of distinct vertices in $G$, we obtain the inequality

$$
|E| \leq \binom{n}{2}.
$$

\[\square\]

Connectivity

Exercise 3. Prove the following theorem:

**Theorem.** Let $H_1, H_2$ be connected subgraphs of a graph $G$ and suppose $H_1$ and $H_2$ have a common vertex. Then $H_1 \cup H_2$ is connected.

**Proof.** By assumption, there is a vertex $v \in V_{H_1} \cap V_{H_2}$. Let $u, w$ be any two vertices in $H_1 \cup H_2$. If $u, w$ are both in $H_1$, then since $H_1$ is connected, there is a walk in $H_1$ from $u$ to $w$, and this walk is also in $H_1 \cup H_2$ since $H_1$ is a subgraph of $H_1 \cup H_2$. A similar argument works when $u, w$ are both in $H_2$. The last case to consider is when $u$ is in $H_1$, while $w$ is in $H_2$. Since $H_1$ is connected, there is a walk $(u, e_1, v_1, \ldots, e_n, v)$ in
**Exercise 4.** Prove the following theorem:

**Theorem.** Let \( H_1, H_2 \) be connected subgraphs of a graph \( G \). Then \( H_1 \cap H_2 \) is connected.

This theorem is false! (Therefore it does not deserve to be called a theorem.) Here is the smallest counterexample:

\[
G = \begin{array}{c}
\bullet & e & w \\
f & \text{..} & \\
\bullet & w
\end{array} \\
H_1 = \begin{array}{c}
\bullet & e & w \\
\text{..} & \text{..} & \\
\bullet & w
\end{array} \\
H_2 = \begin{array}{c}
\bullet & f & w \\
\text{..} & \text{..} & \\
\bullet & w
\end{array}
\]

Clearly \( H_1 \) and \( H_2 \) are connected subgraphs of \( G \), but \( H_1 \cap H_2 = \begin{array}{c}
\bullet & w \\
\text{..} & \text{..} & \\
\bullet & w
\end{array} \), which is not a connected subgraph of \( G \).

**Comment:** It is tempting to think that this theorem can be proven, because you can find walks in both \( H_1 \) and \( H_2 \) between any two vertices of \( H_1 \cap H_2 \). The trouble is that these two walks can be very different, and there is no way to combine them into a walk in \( H_1 \cap H_2 \). For instance, in the counterexample above, \((v, e, w)\) is a walk in \( H_1 \) from \( v \) to \( w \), and \((v, f, w)\) is a walk in \( H_2 \) from \( v \) to \( w \), but there is no way to combine these two walks into a walk in \( H_1 \cap H_2 \).