Section 1.7

General comments:

- A “2” means you need to work on your proof writing style and/or the logic of your arguments. A “3” means your proof writing is overall pretty good. Writing proofs is a skill that will be crucial all semester long, so make sure you are comfortable writing proofs!

- State the theorems!

- State clearly where any variables are coming from.

- You don’t need to explain definitions (like even and odd).

- You don’t need to write out easy intermediate steps when doing arithmetic (like cubing $2k + 1$). That’s what your scratch paper is for!

- Write in full sentences! If you are writing precisely and skipping easy arithmetic, you probably won’t need to write that much. Notice how short my proofs are below.

- Don’t cram your writing into a tiny space! Mathematics needs room to breathe!

- Test your proof on a classmate, check in with me, and/or compare with the solutions in the textbook (for odd numbered problems).

- If you are willing, please type your solutions! All mathematicians I know use the free typesetting language \LaTeX. It takes a bit of work to learn, but it is worth it if you plan on continuing with mathematics, and makes editing proofs much easier.

Exercise 1.

Theorem. The sum of two odd integers is even.

Proof. Let $n$ and $m$ be odd integers. Then there exist integers $k$ and $l$ such that $n = 2k + 1$ and $m = 2l + 1$. Thus $n + m = 2(k + l + 1)$, which is even.

Specific comments:
• The two odd integers can be different! That’s why I used two variables, \( n \) and \( m \).

• It is okay (in fact, recommended!) to suppress simple arithmetic, like the computation
  \( (2k + 1) + (2l + 1) = 2k + 2l + 2 = 2(k + l + 1) \).

**Exercise 9.**

**Theorem.** The sum of a rational number and an irrational number is irrational.

**Proof.** Let \( x \) be rational and \( y \) be irrational. Assume for contradiction that \( x + y \) is rational. Then there exist integers \( a \) and \( b \) with \( b \neq 0 \) such that \( x = \frac{a}{b} \), and integers \( c \) and \( d \) with \( d \neq 0 \) such that \( x + y = \frac{c}{d} \). But then \( y = \frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd} \), which is rational since \( bd \neq 0 \). This contradicts the fact that \( y \) is irrational.

Specific comments:

• Remember that for a proof by contradiction, you are adding the negation of the conclusion to the premises, and try to derive a contradiction. The “assume for contradiction” sentence informs the reader that you are assuming the negation of the conclusion.

• Don’t bother “creating” a new real number \( z \) to represent the sum \( x + y \). It is already quite clear that \( x + y \) is a real number!

**Exercise 15.**

**Theorem.** If \( x + y \geq 2 \), where \( x, y \in \mathbb{R} \), then \( x \geq 1 \) or \( y \geq 1 \).

**Proof.** I will prove the contrapositive, namely that if \( x < 1 \) and \( y < 1 \), then \( x + y < 2 \). Suppose \( x < 1 \) and \( y < 1 \). Adding \( y \) to both sides of the inequality \( x < 1 \), and adding 1 to both sides of the inequality \( y < 1 \), we see that \( x + y < 1 + y < 1 + 1 = 2 \), as we wanted to show.

Specific comments:

• The negation of “\( x \geq 1 \) or \( y \geq 1 \)” is “\( x < 1 \) and \( y < 1 \)”.

• You can also just “add” the inequalities \( x < 1 \) and \( y < 1 \) to get \( x + y < 2 \).

**Exercise 30.**

**Theorem.** For \( a, b \in \mathbb{R} \), the following statements are equivalent:

(i) \( a < b \);

(ii) \( a < \frac{a+b}{2} \);

(iii) \( \frac{a+b}{2} < b \).

**Proof.** It suffices to prove (i) \( \implies \) (ii), (ii) \( \implies \) (iii), and (iii) \( \implies \) (i). For (i) \( \implies \) (ii), suppose \( a < b \). Adding \( a \) to both sides and dividing by 2, we get \( a < \frac{a+b}{2} \). For (ii) \( \implies \) (iii), assume \( a < \frac{a+b}{2} \). Adding \( \frac{b-a}{2} \) to both sides gives \( \frac{a+b}{2} < b \). Finally, for (iii) \( \implies \) (i), suppose \( \frac{a+b}{2} < b \). Multiplying by 2 and subtracting \( b \), we get \( a < b \).
Specific comments:

- “Equivalent” means that if any one of the statements is true, then all the others must also be true, and if any one is false, then all the others must be false. This means showing that (i) $\iff$ (ii), (i) $\iff$ (iii), and (ii) $\iff$ (iii). But in fact it is enough to show a “loop of implications” (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i) that contains all of the statements, because the truth of any one statement “runs around the loop” to establish the truth of all the others.

Additional Problem

Lemma. If $n$ is an integer and $n^3$ is even, then $n$ is even.

Proof. I will prove the contrapositive. Suppose $n$ is odd. Then there exists $k \in \mathbb{Z}$ such that $n = 2k + 1$. Thus $n^3 = (2k + 1)^3 = 2(4k^3 + 6k^2 + 3k) + 1$, so $n^3$ is odd. \qed

Theorem. $\sqrt[3]{2}$ is irrational.

Proof. Assume for contradiction that $\sqrt[3]{2}$ is rational. Then there exist $a, b \in \mathbb{Z}$ with $b \neq 0$ such that $a$ and $b$ have no common factors and $\sqrt[3]{2} = \frac{a}{b}$. Cubing both sides and clearing denominators, we get $a^3 = 2b^3$, so $a^3$ is even, which by the lemma implies that $a$ is even. Thus there is a $k \in \mathbb{Z}$ such that $a = 2k$, which implies $2b^3 = 8k^3$, so $b^3 = 2(2k^3)$. Therefore $b^3$ is even, so the lemma shows that $b$ is even. But we have shown that both $a$ and $b$ are even, contradicting the fact that $a$ and $b$ have no common factors. \qed