

Mathemagic of Sea Ice, Semiconductors, and Harry Potter

Kenneth Golden
University of Utah

Reid Lecture
CSU San Marcos

April 17, 2025

sea ice



Golden

semiconductor



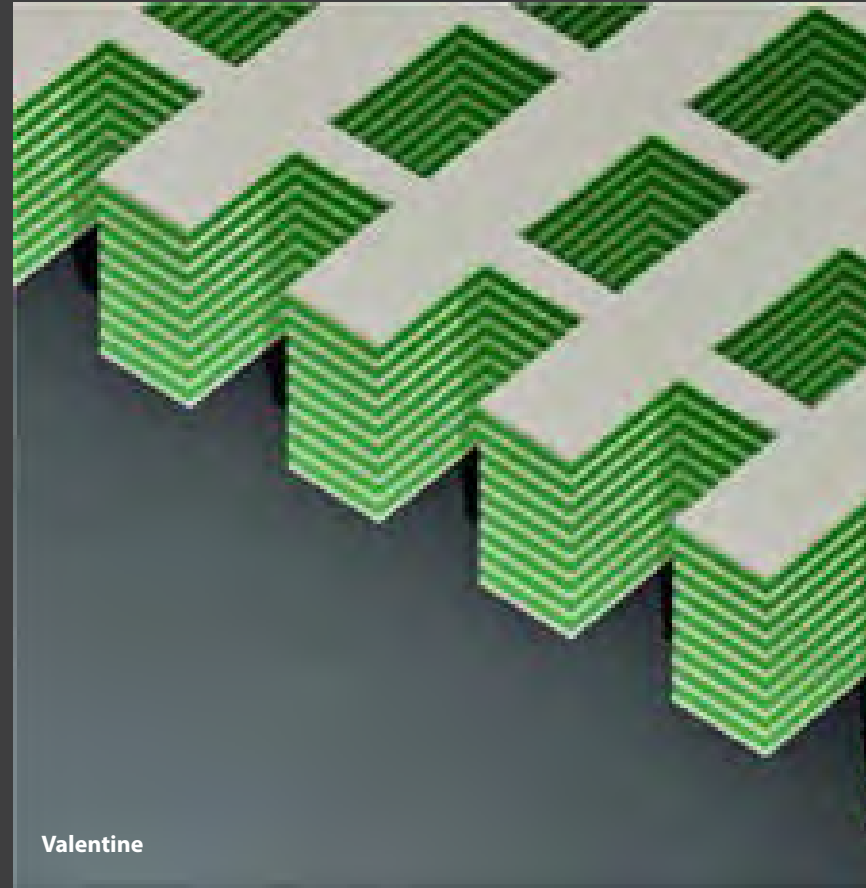
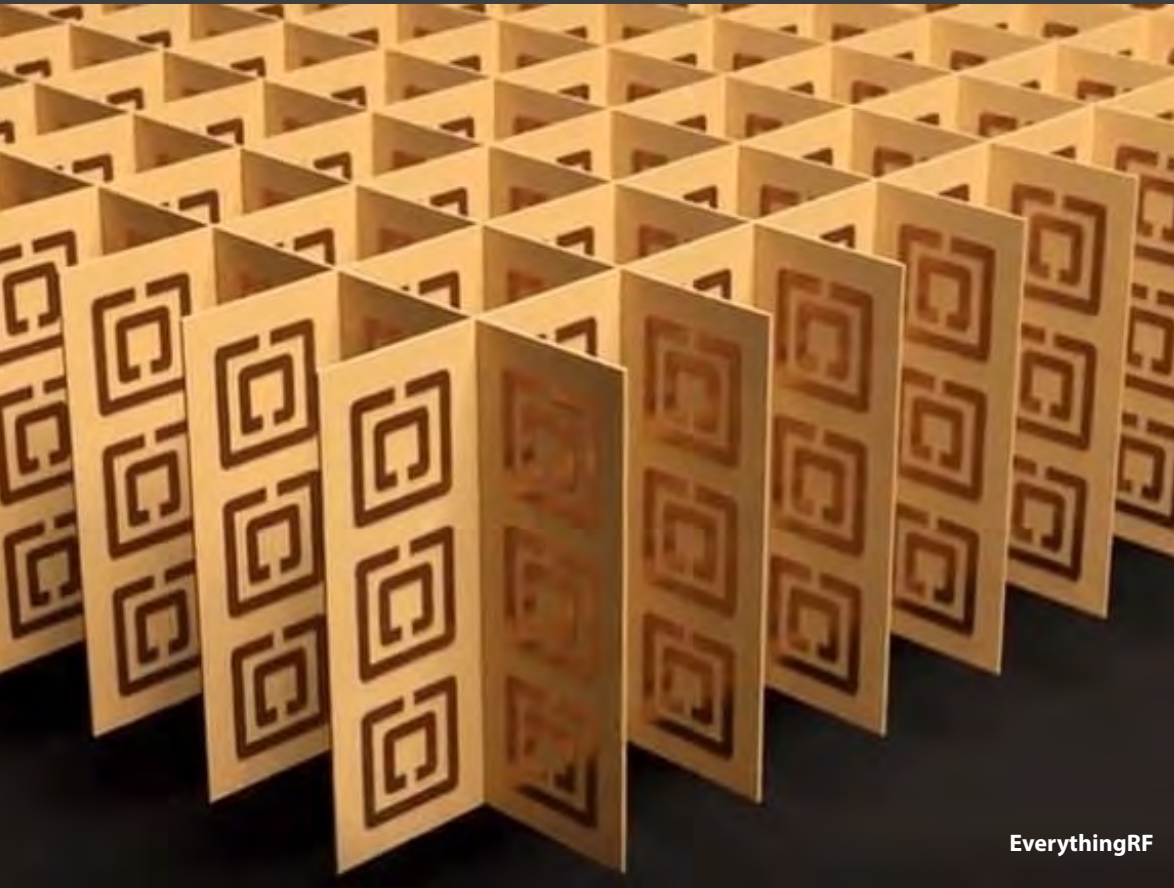
Shutterstock



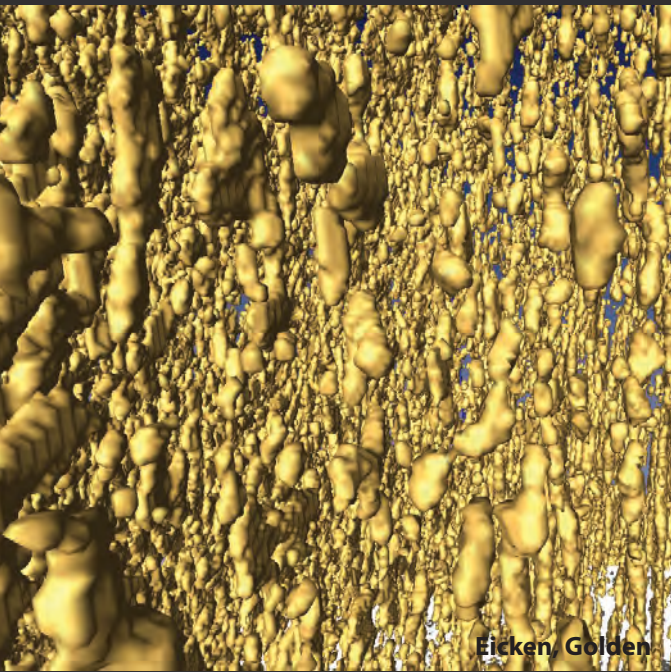
Warner Brothers

invisibility cloak

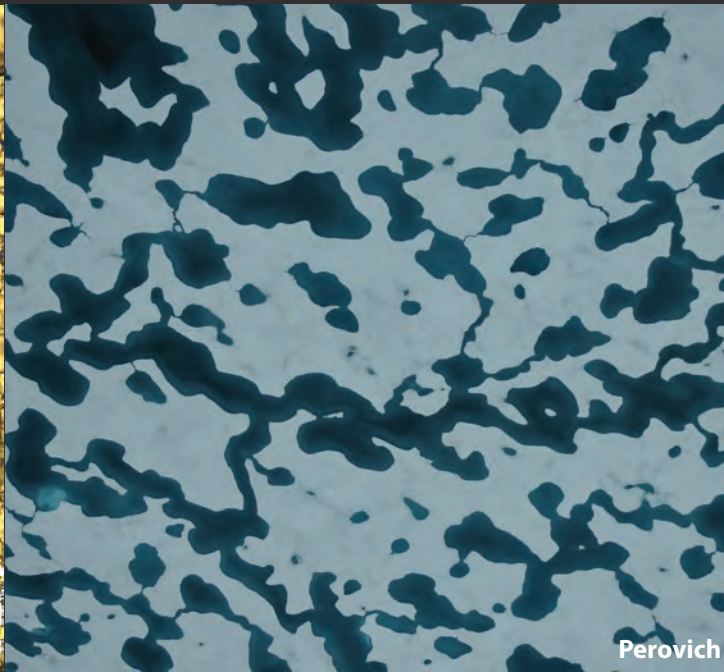
metamaterials



Sea ice is a multiscale composite material.



micro



meso

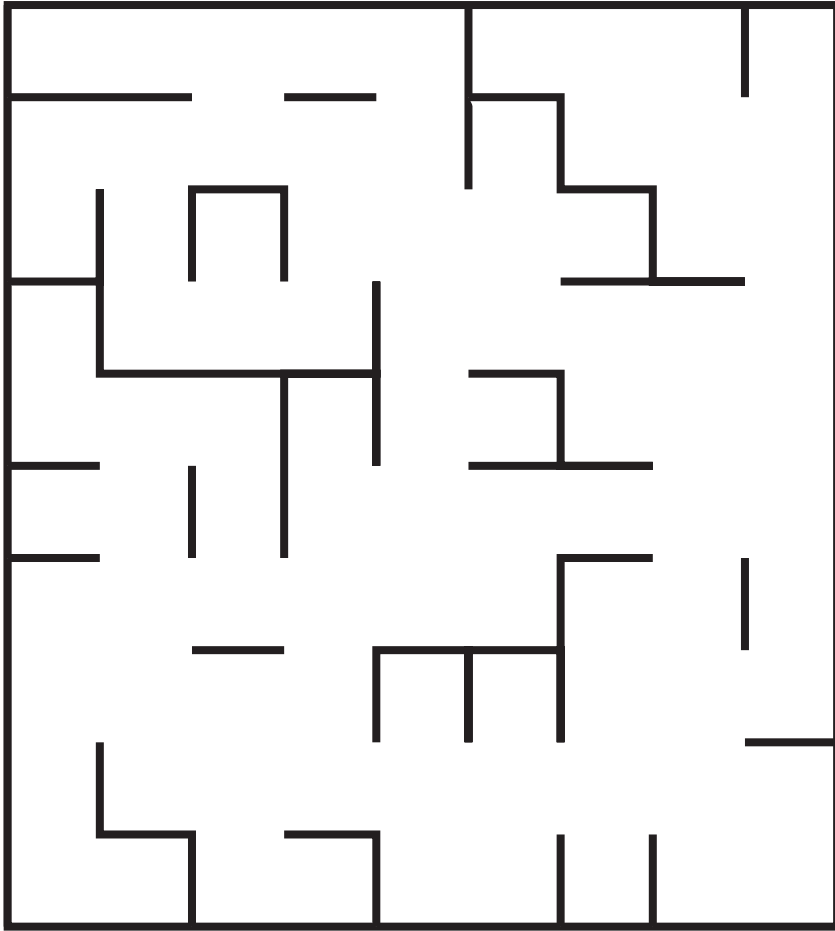


macro

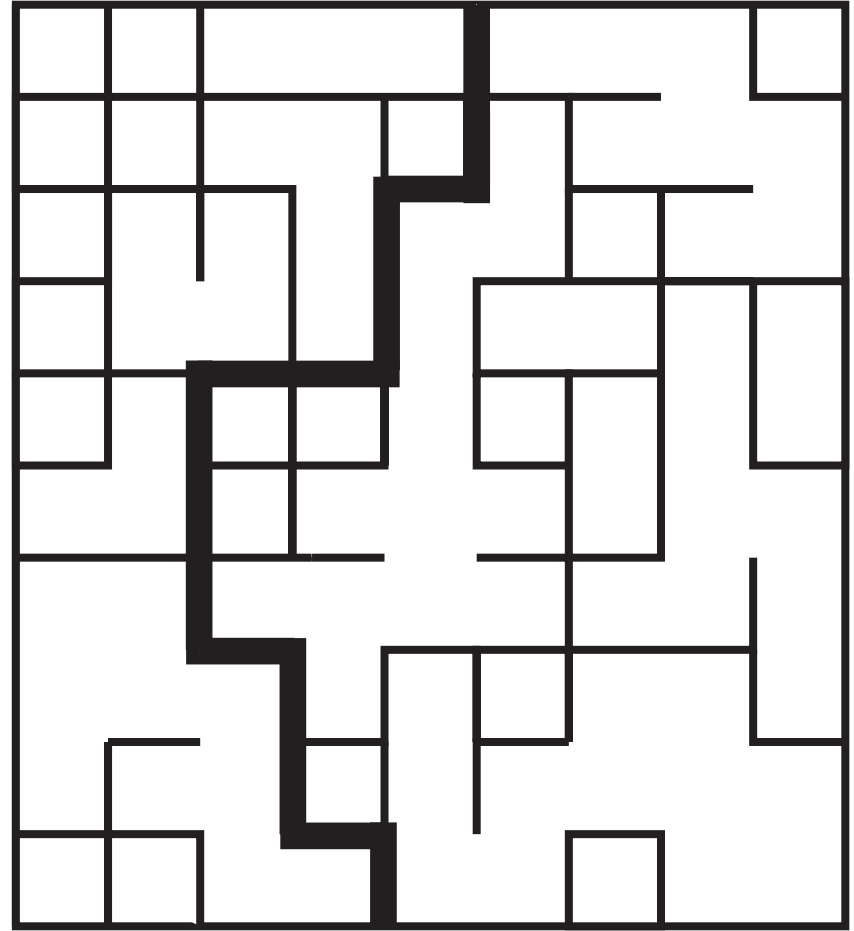


Weddell Sea, Antarctica

percolation model



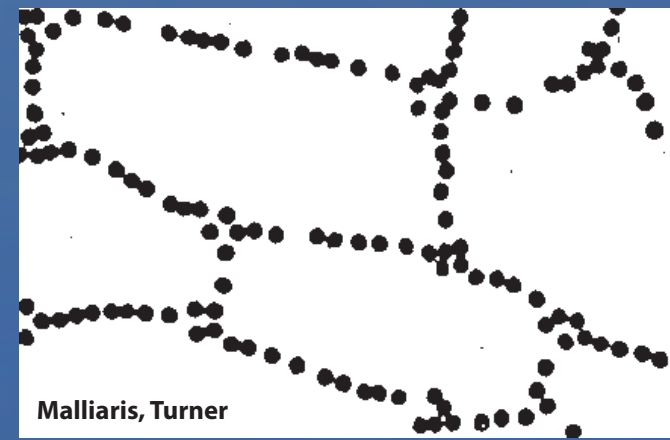
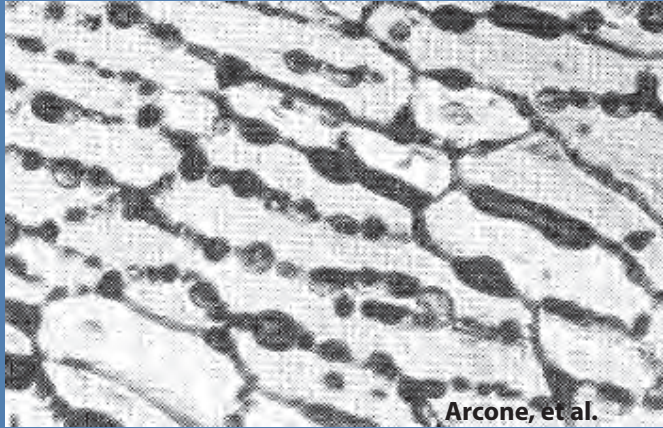
$$p = 1/3$$



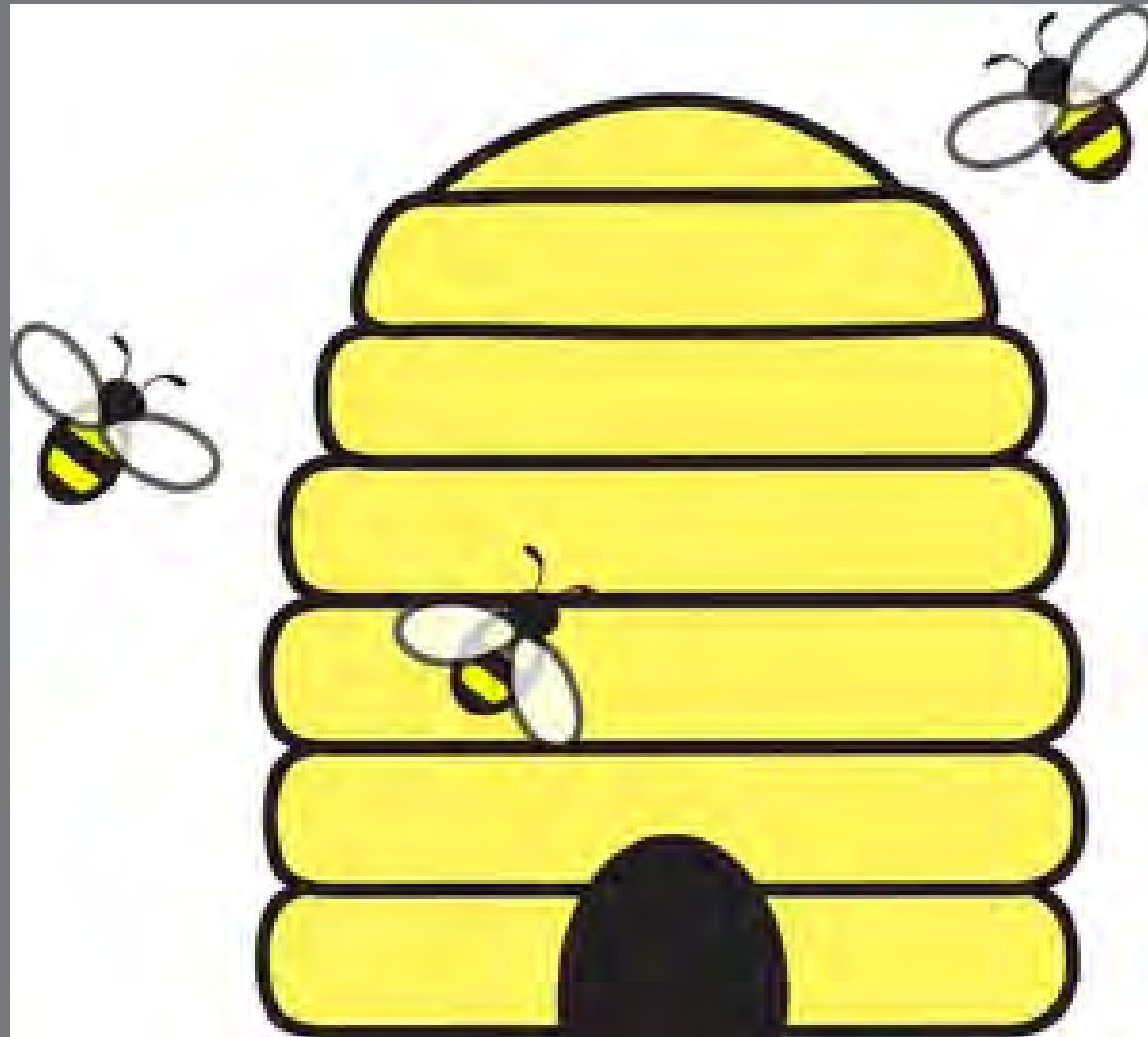
$$p = 2/3$$

$$p_c = ?$$

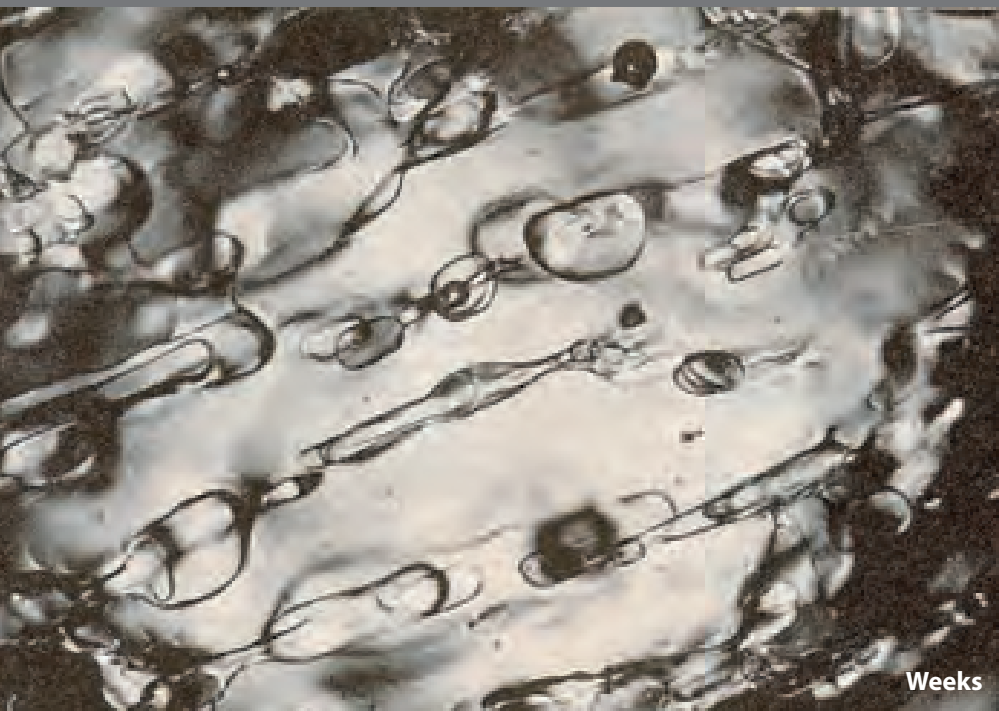
stealth



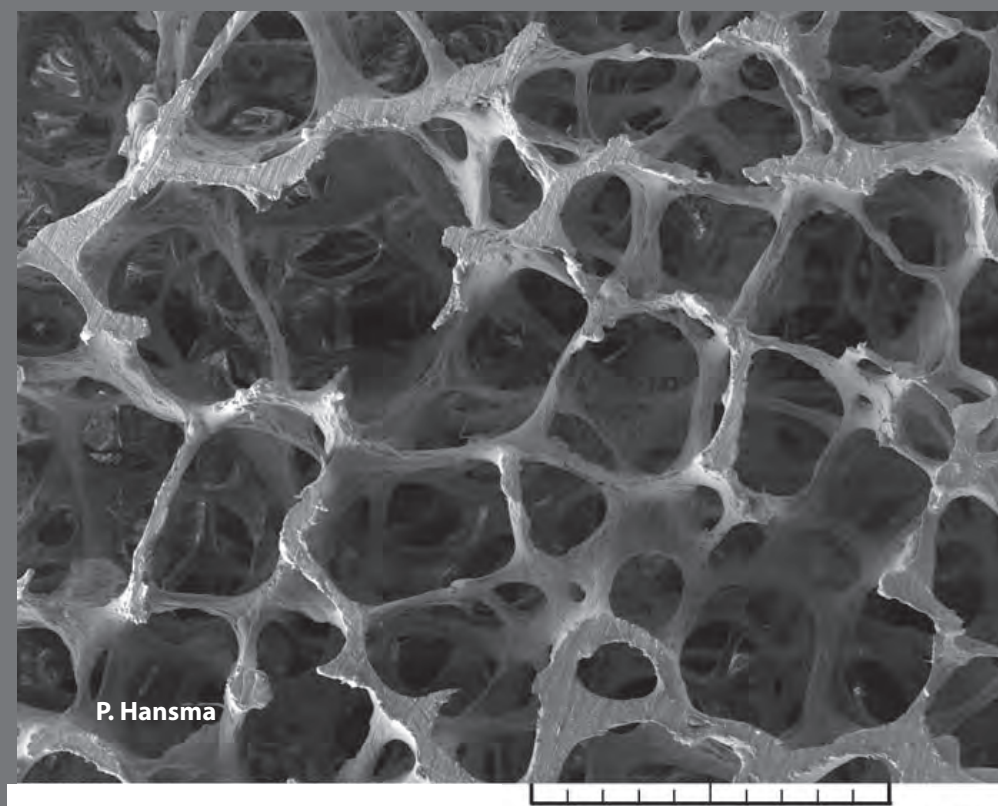
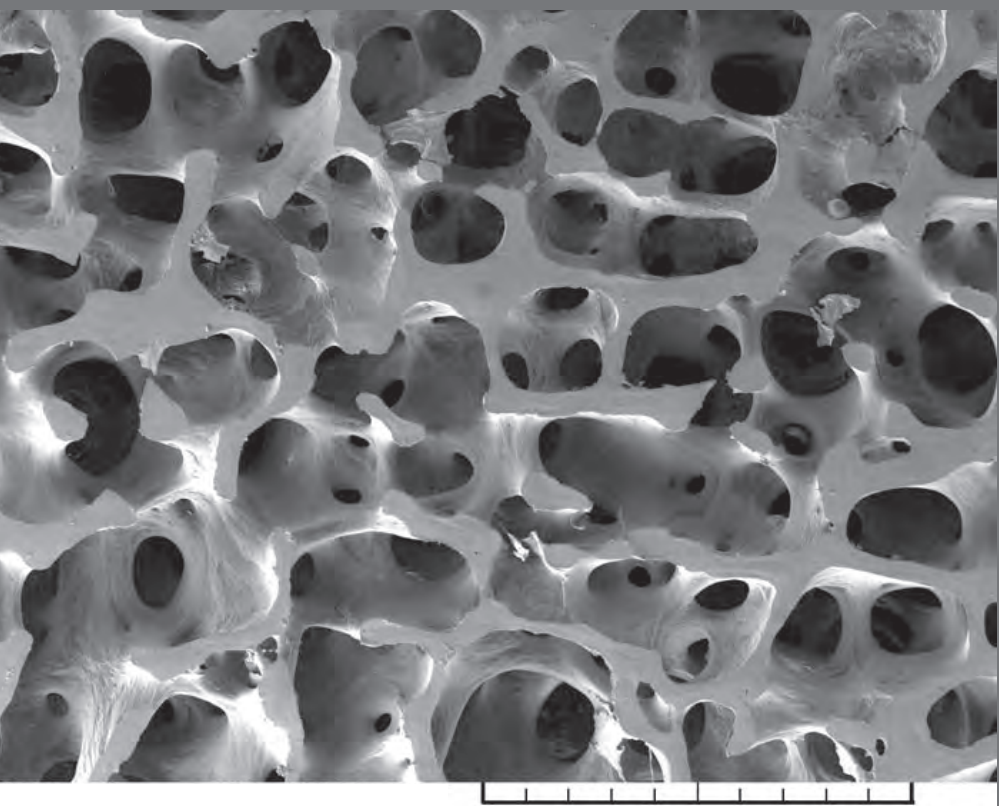
cross-pollination



sea ice



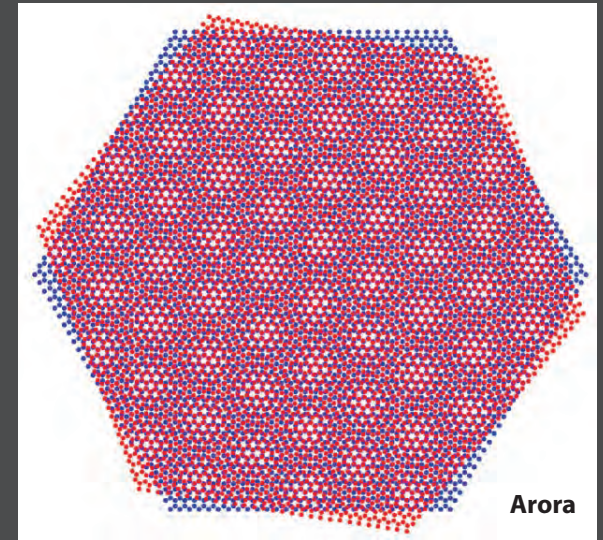
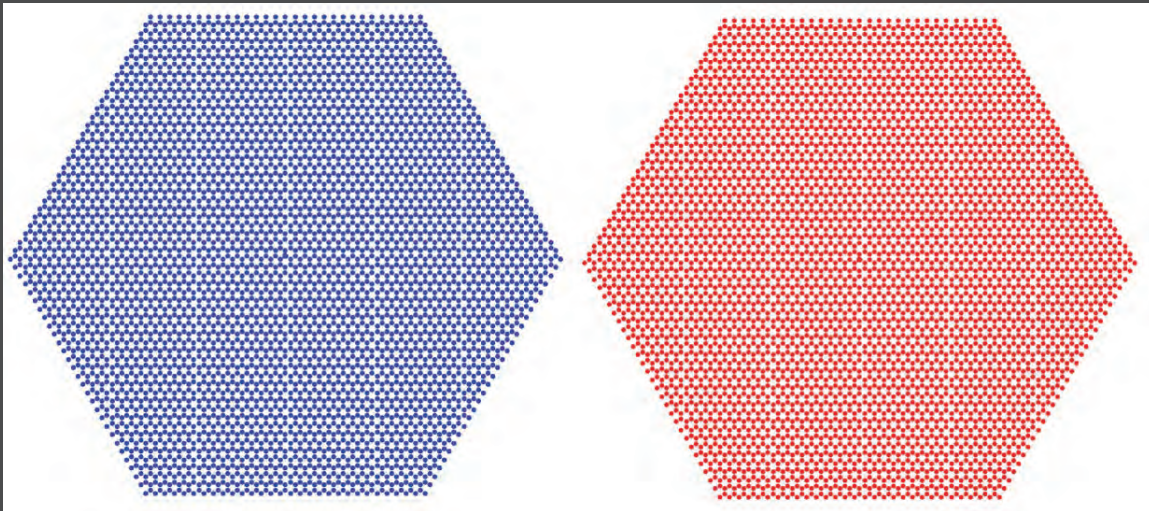
human bone



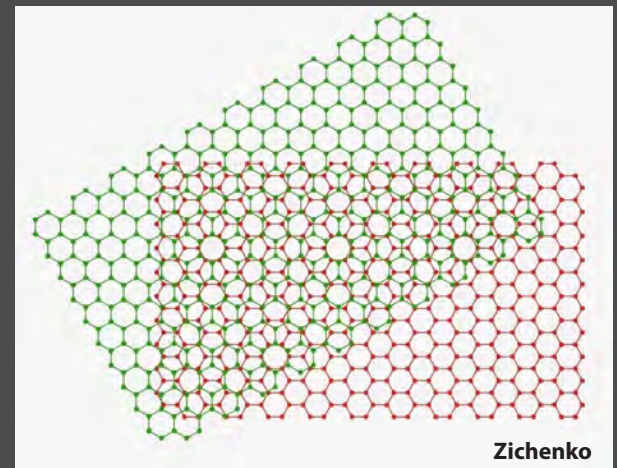
waves on frozen seas



twisted bilayer graphene



magic angle



Arctic Mathpedition 2024



Reimer

Golden

SEA ICE covers ~12% of Earth's ocean surface

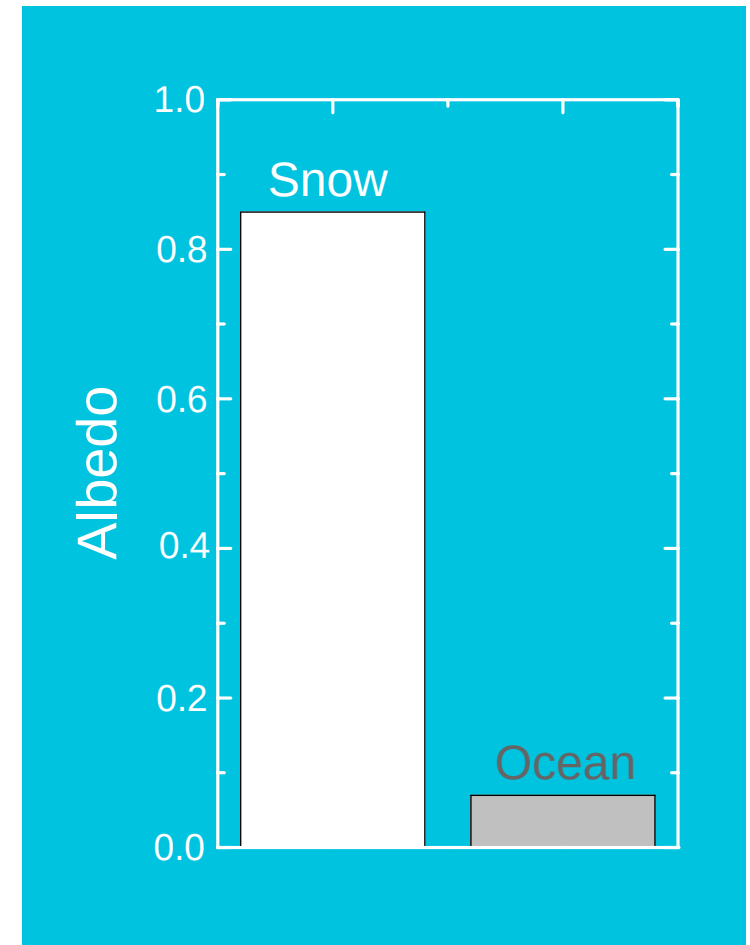
- boundary between ocean and atmosphere
- mediates exchange of heat, gases, momentum
- global ocean circulation
- hosts rich ecosystem
- indicator of **climate change**



polar ice caps critical to global climate in reflecting incoming solar radiation



white snow and ice
reflect

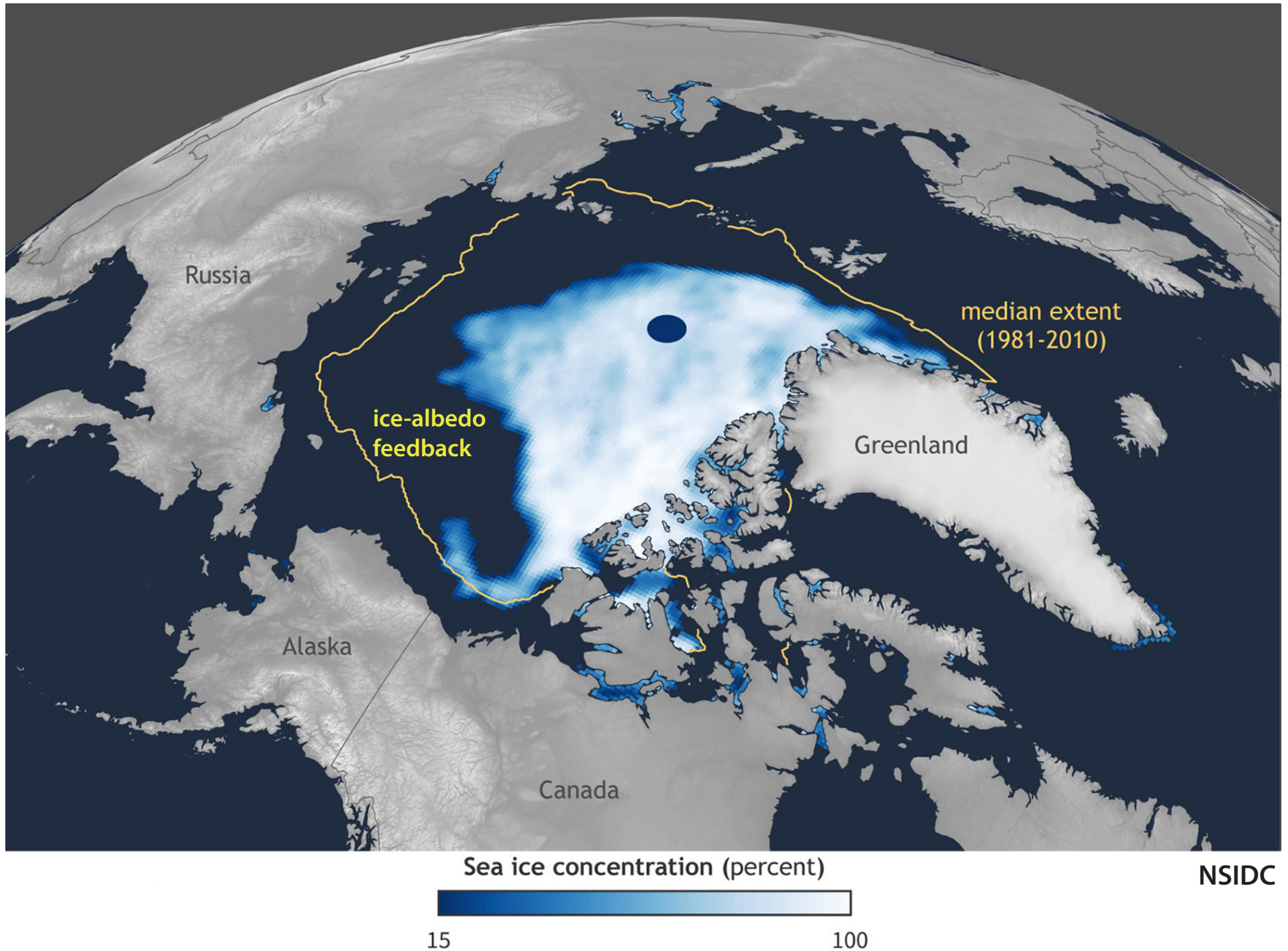


dark water and land
absorb

$$\text{albedo } \alpha = \frac{\text{reflected sunlight}}{\text{incident sunlight}}$$

Arctic sea ice extent

September 15, 2020



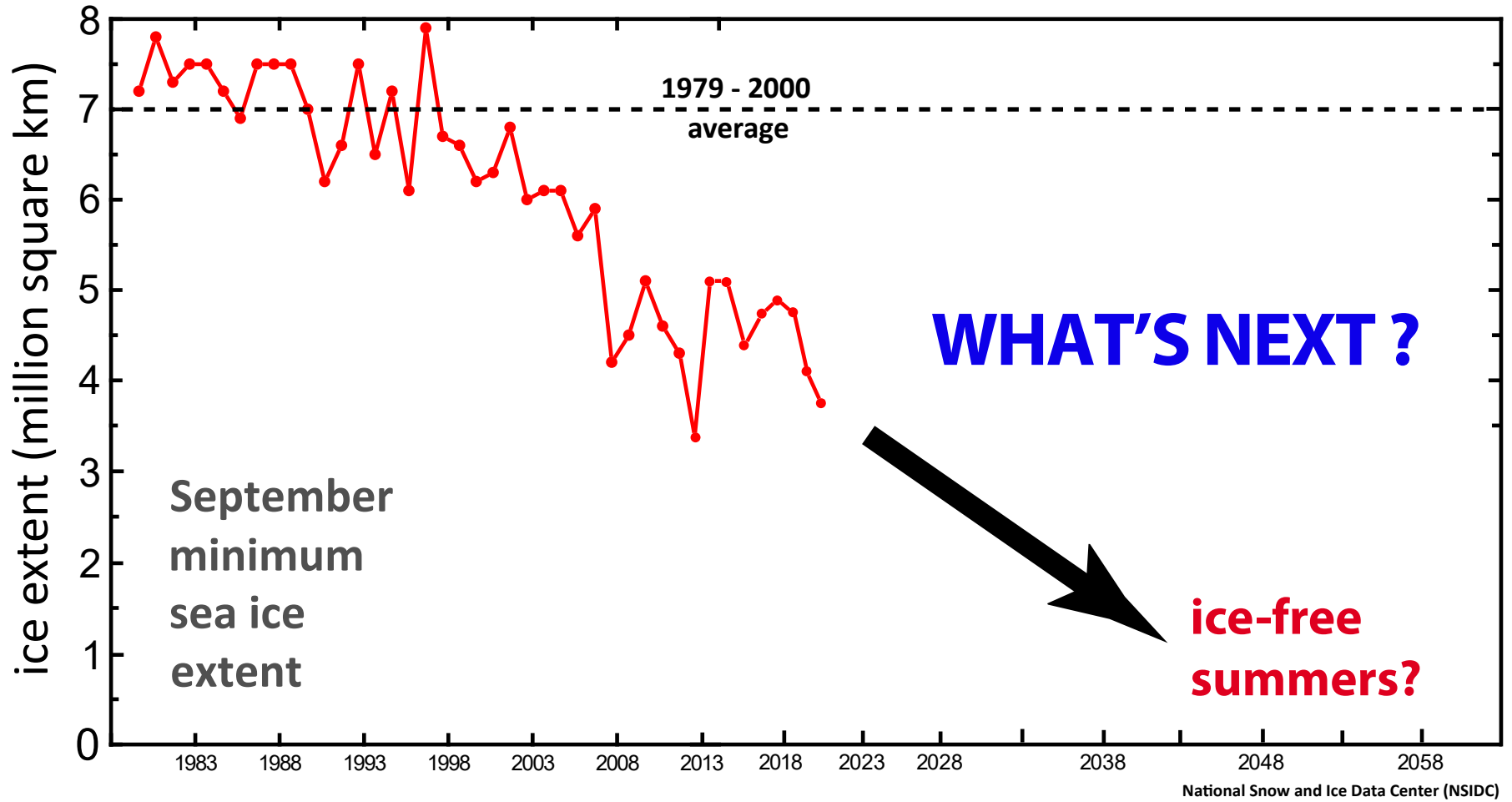


*recent losses
in comparison to
the United States*

Perovich



ARCTIC summer sea ice loss



predictions require lots of math modeling

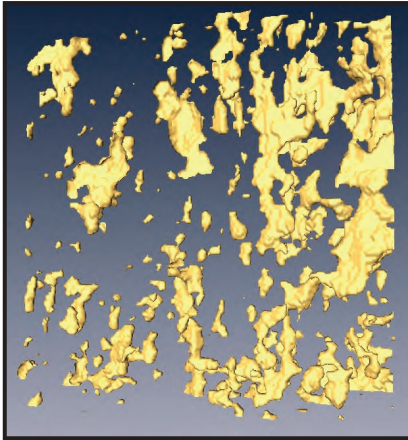
Sea Ice is a Multiscale Composite Material

microscale

brine inclusions

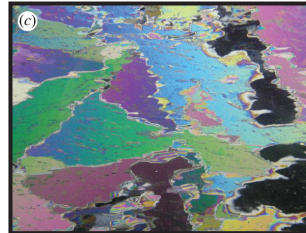
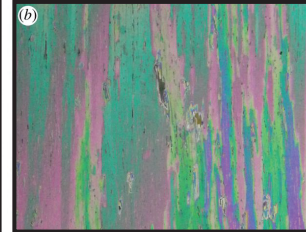
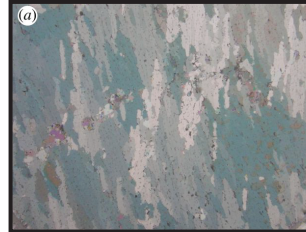


Weeks & Assur 1969



H. Eicken
Golden et al. GRL 2007

polycrystals

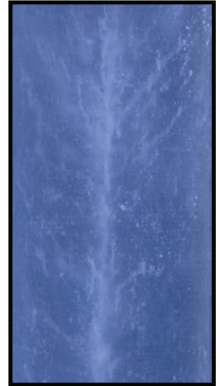


Gully et al. Proc. Roy. Soc. A 2015

brine channels



D. Cole



K. Golden

millimeters

centimeters

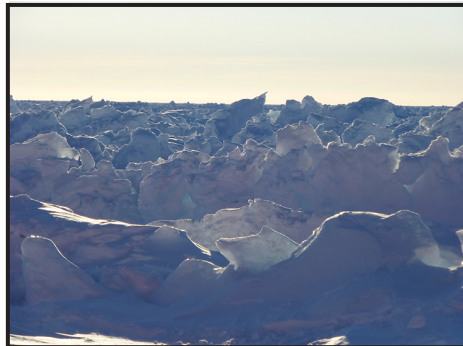
mesoscale

Arctic melt ponds



K. Frey

Antarctic pressure ridges



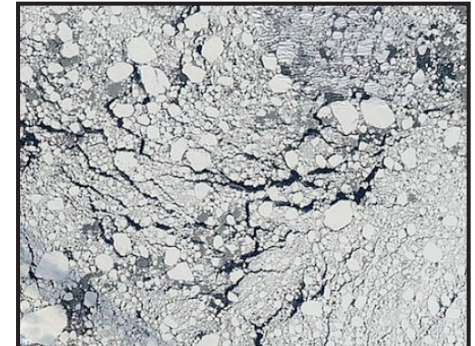
K. Golden

sea ice floes



J. Weller

sea ice pack



NASA

meters

kilometers

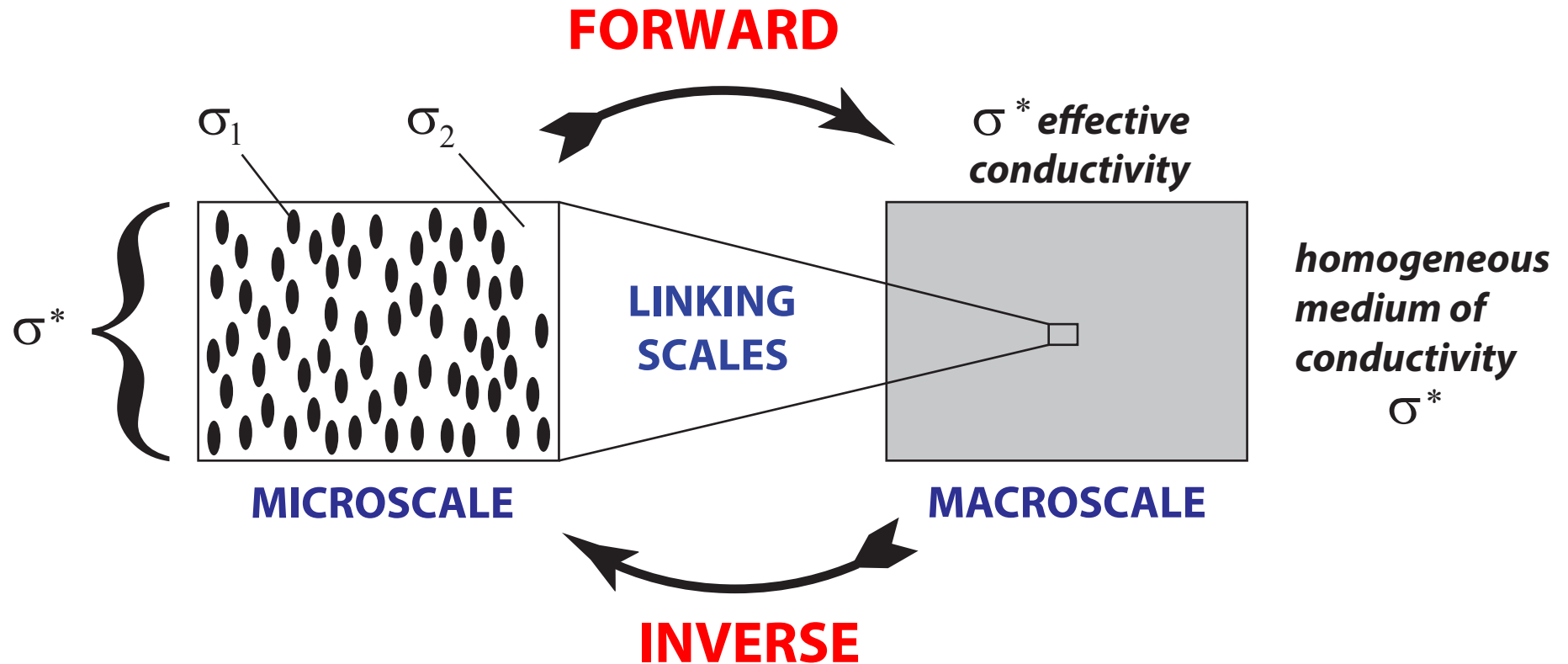
macroscale

Central theme:

How do we use “small scale” information to find effective behavior on larger scales relevant to climate and ecological models?

**OBJECTIVE: advance how sea ice is represented in climate models
improve projections of fate of SEA ICE and its ECOSYSTEMS**

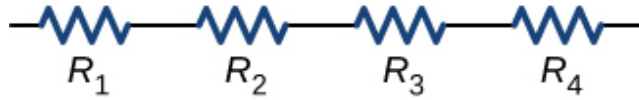
HOMOGENIZATION for Composite Materials



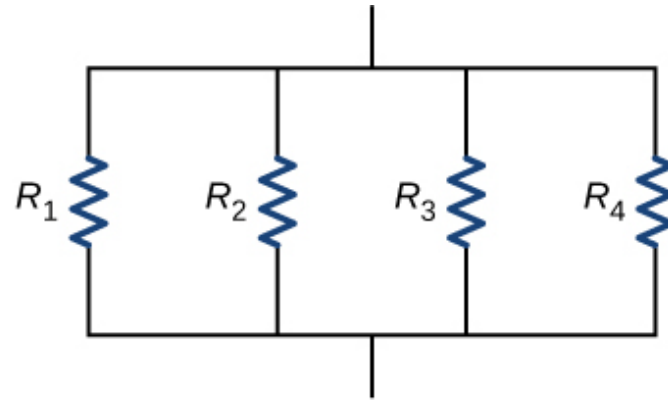
Maxwell 1873, Einstein 1906

Wiener 1912, Hashin and Shtrikman 1962

resistors in series



resistors in parallel



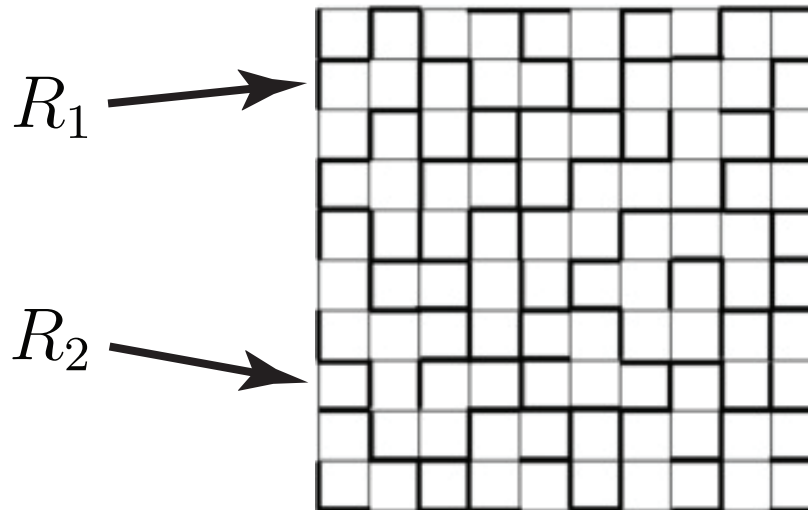
equivalent resistance

$$R_{eq} = R_1 + R_2 + R_3 + R_4$$

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}$$

**random
resistor network**

**percolation
model**



$$R_{eq} = ??$$

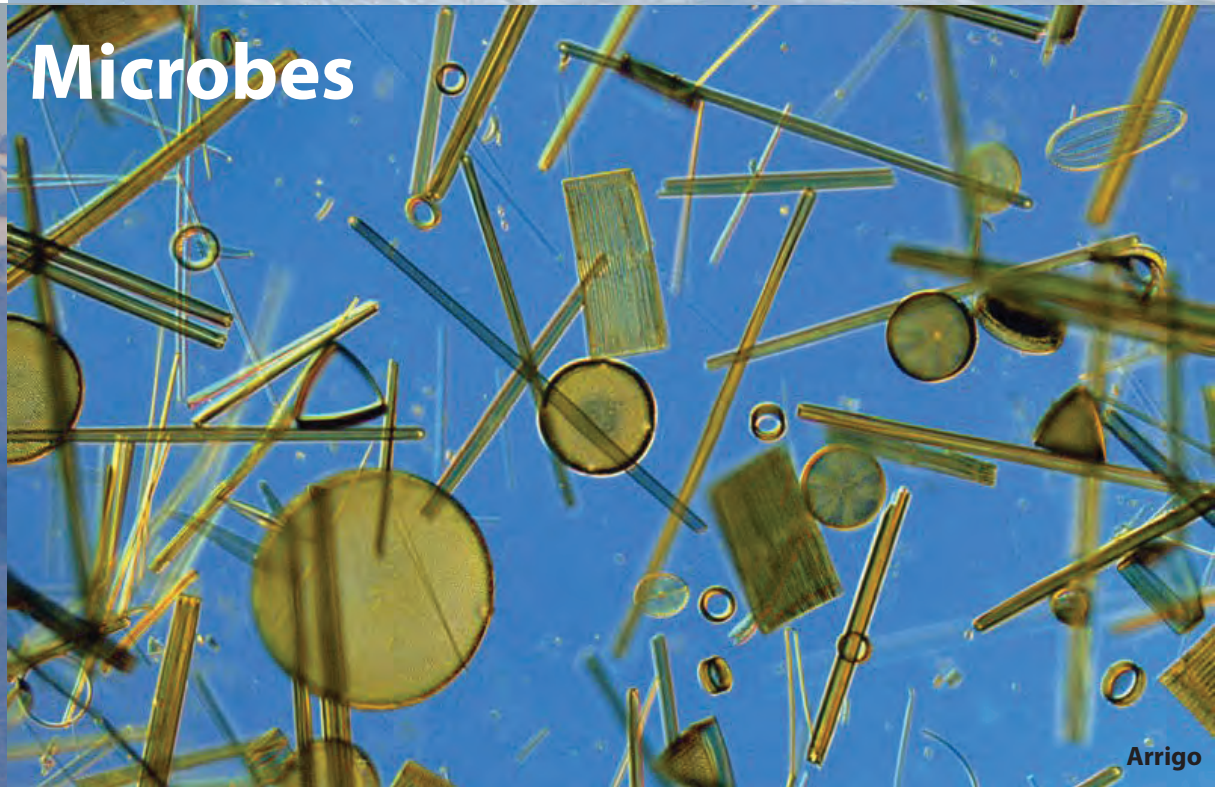
infinite lattice ?

discrete
homogenization

Polar Ecology and the Physics of Sea Ice

How do sea ice properties affect the life it hosts?

How does life in and on sea ice affect its physical properties?



Tour a few examples of multiscale modeling of physical and biological processes in the sea ice system.

microscale

mesoscale

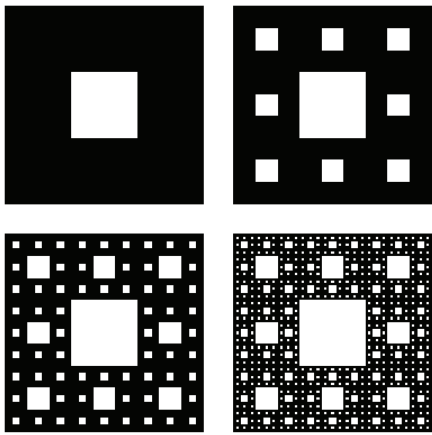
macroscale

Take-away for mathematicians and physicists - our sea ice studies lead us into:

**spectral analysis, random matrix theory, topological data analysis,
UQ, anomalous diffusion, dynamical systems**

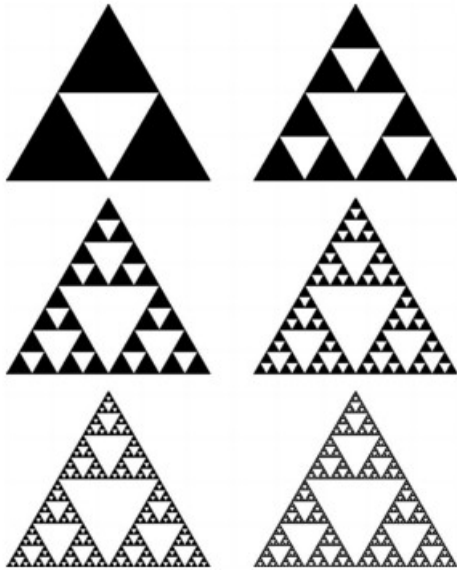
**percolation, Anderson localization, mushy layers, phase transitions
semiconductors, Ising models, quasicrystals**

+ fractal geometry

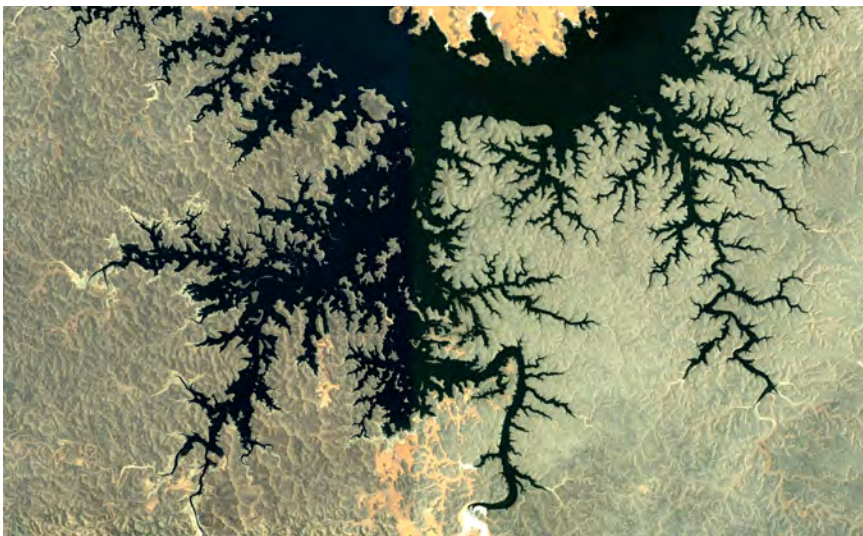


fractals

self-similar structure
non-integer dimension

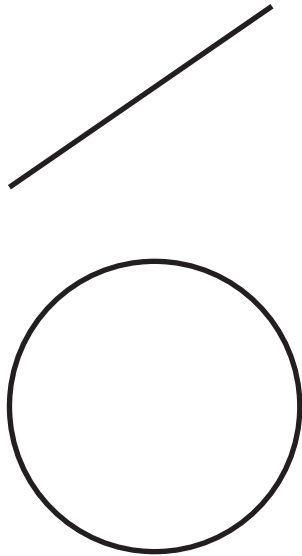


$$D = \frac{\log 3}{\log 2} = 1.585...$$



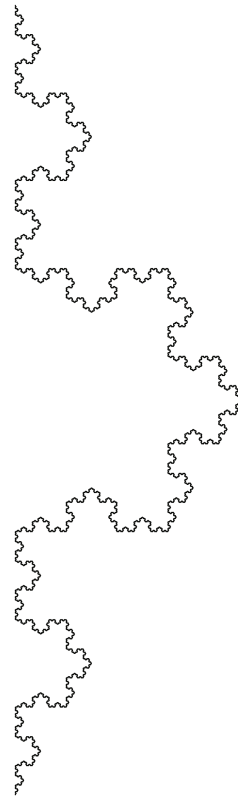
fractal curves in the plane

they wiggle so much that their dimension is >1



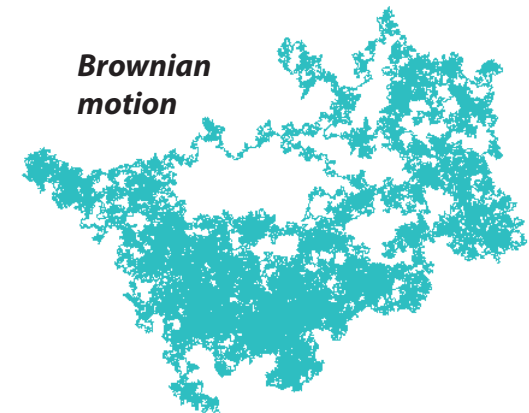
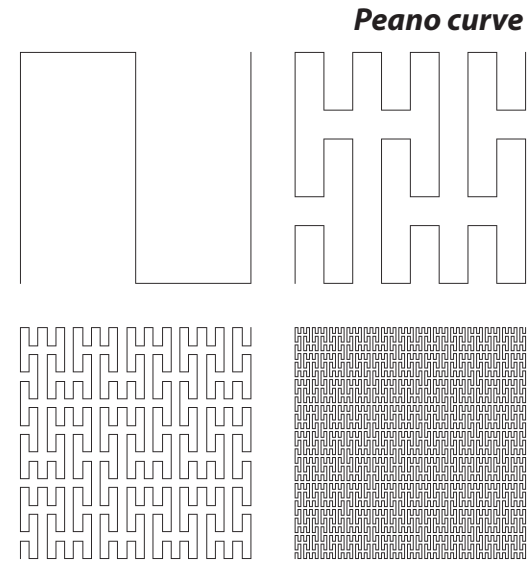
simple curves

$D = 1$



Koch snowflake

$D = 1.26$



Brownian motion

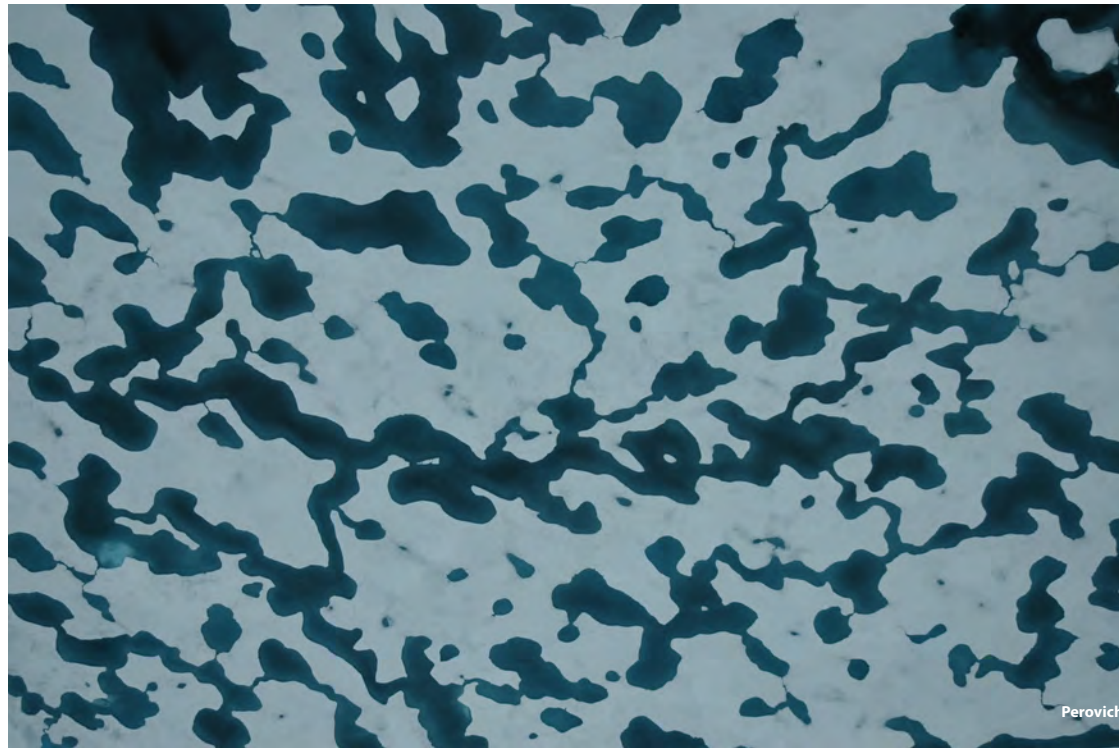
space filling curves

$D = 2$

microscale

fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

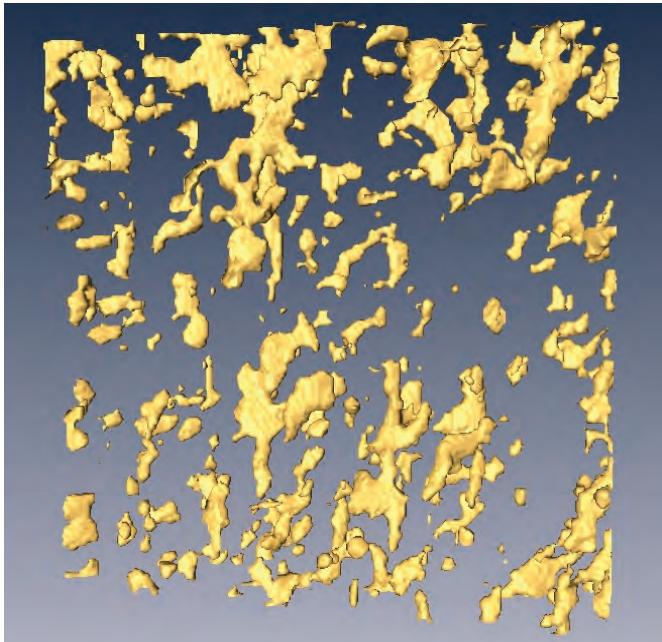
evolution of Arctic melt ponds and sea ice albedo



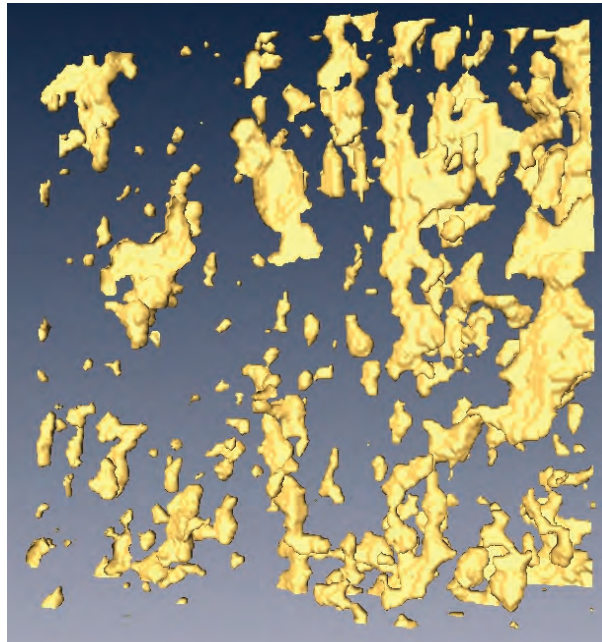
nutrient flux for algal bloom



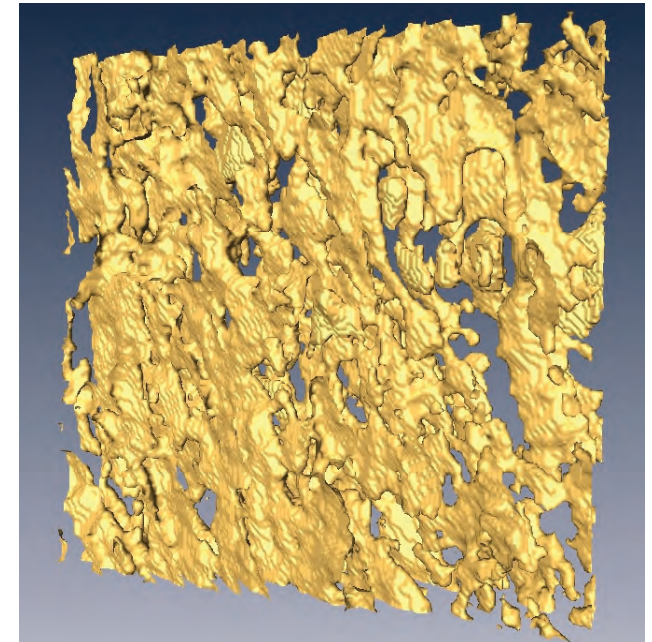
brine volume fraction and **connectivity** increase with temperature



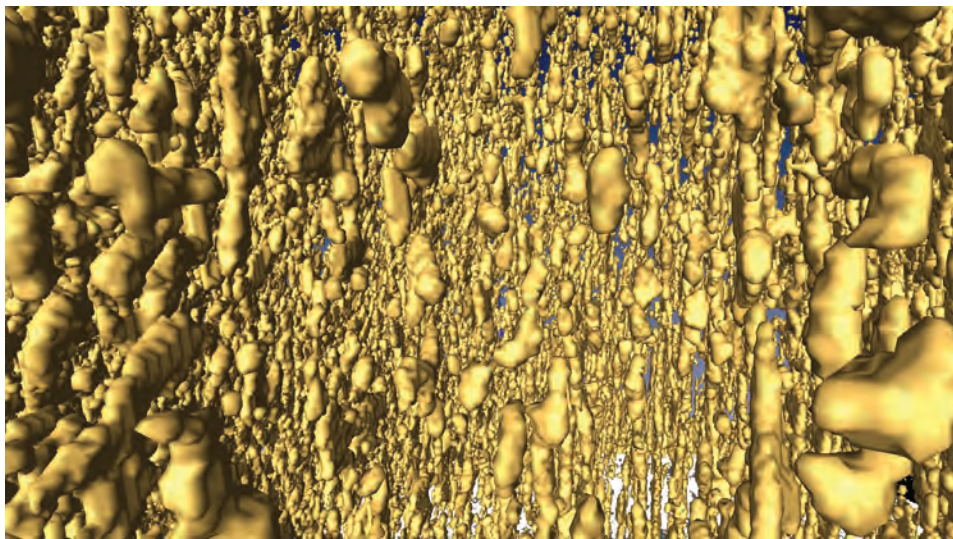
$T = -15\text{ }^{\circ}\text{C}$, $\phi = 0.033$



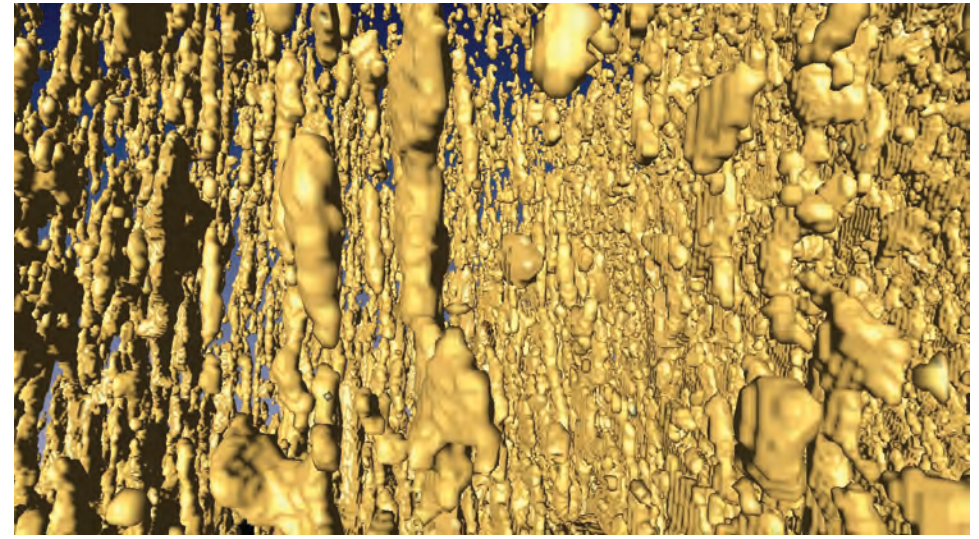
$T = -6\text{ }^{\circ}\text{C}$, $\phi = 0.075$



$T = -3\text{ }^{\circ}\text{C}$, $\phi = 0.143$



$T = -8\text{ }^{\circ}\text{C}$, $\phi = 0.057$



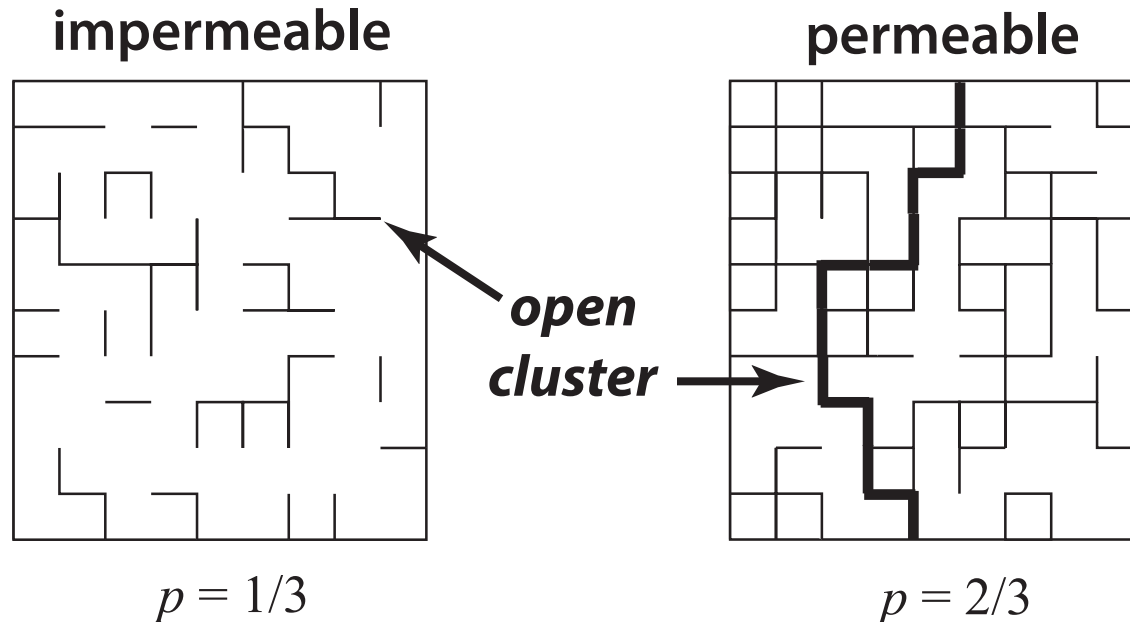
$T = -4\text{ }^{\circ}\text{C}$, $\phi = 0.113$

X-ray tomography for brine in sea ice

Golden et al., *Geophysical Research Letters*, 2007

percolation theory

probabilistic theory of connectedness



bond $\longrightarrow$ open with probability p
closed with probability $1-p$

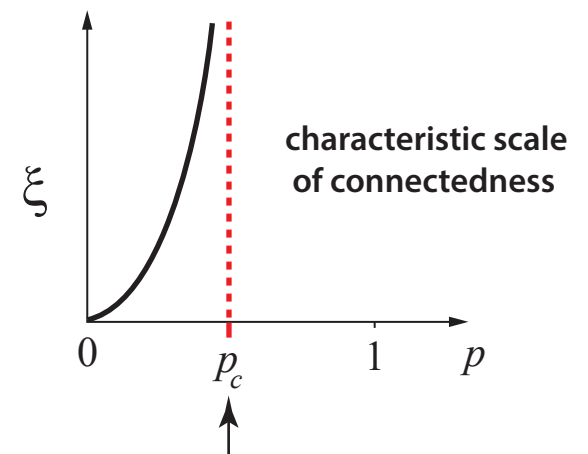
percolation threshold

$$p_c = 1/2 \quad \text{for } d = 2$$

smallest p for which there is an infinite open cluster

correlation length

development of long range order



$$\xi(p) \sim |p - p_c|^{-\nu} \quad p \rightarrow p_c$$

ν universal: depends only on d

p_c depends on type of lattice and d

Critical behavior of fluid transport in sea ice

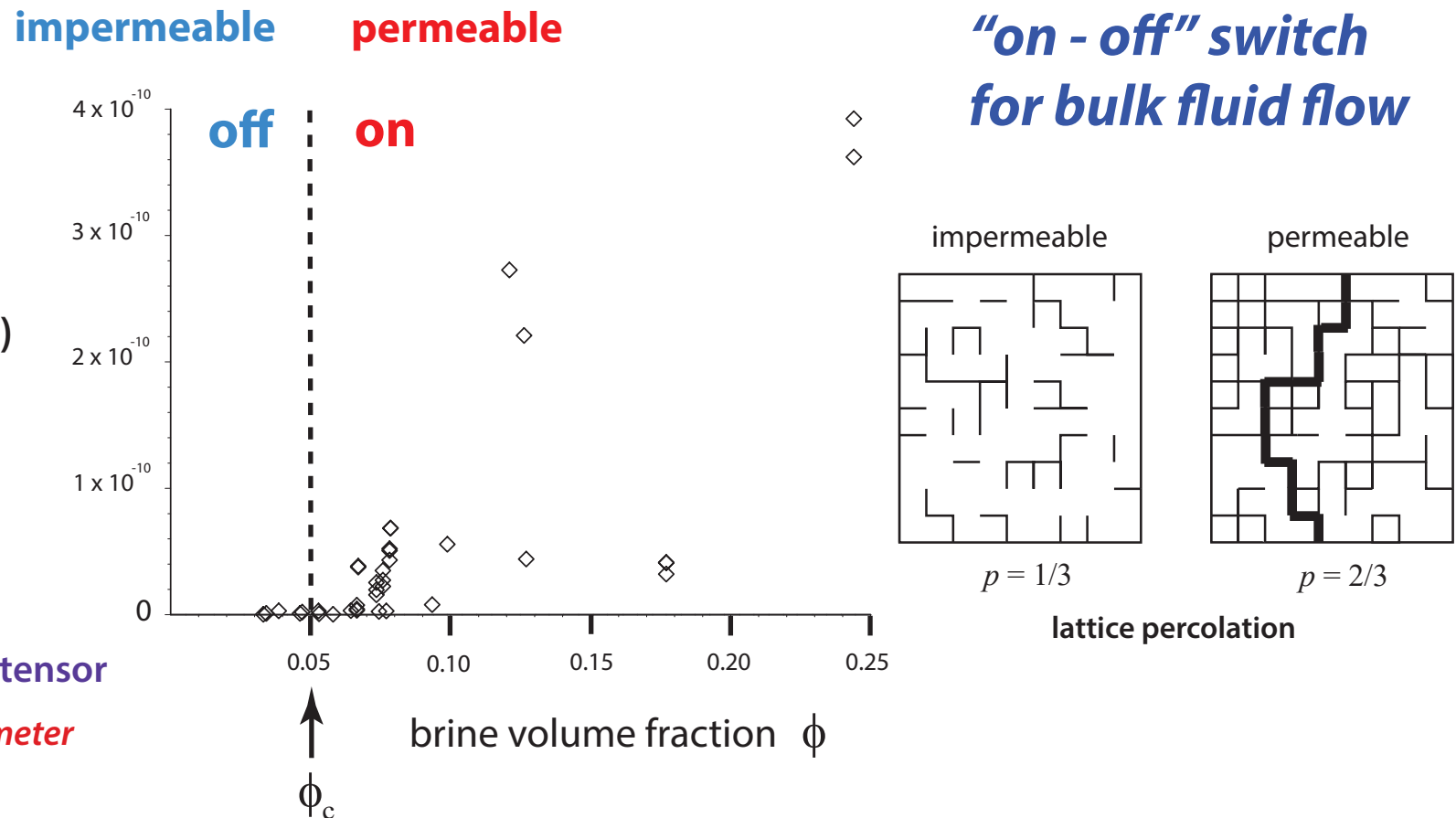
Arctic field data

vertical fluid permeability k (m^2)

Darcy's Law

$$\mathbf{v} = -\frac{\mathbf{k}}{\eta} \nabla p$$

$\mathbf{k}$ = fluid permeability tensor
homogenized parameter



PERCOLATION THRESHOLD $\phi_c \approx 5\% \longleftrightarrow T_c \approx -5^\circ \text{C}, S \approx 5 \text{ ppt}$

RULE OF FIVES

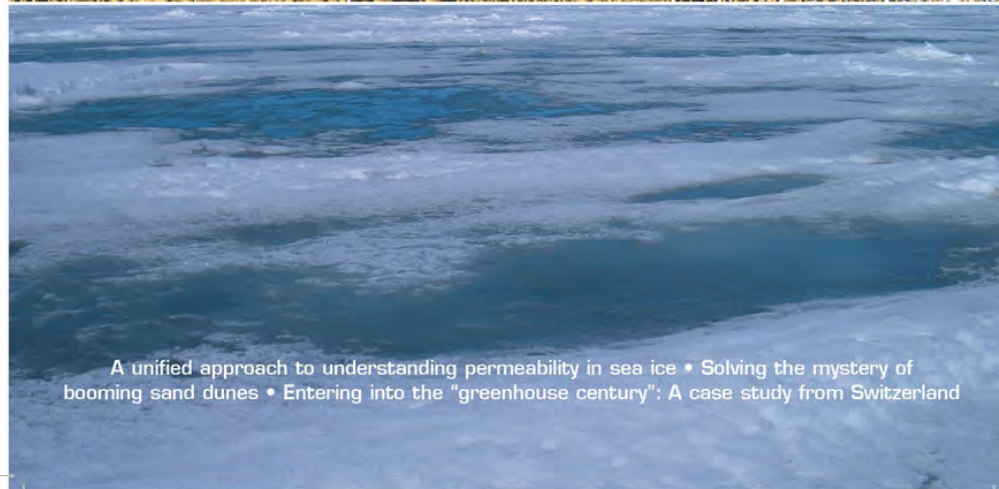
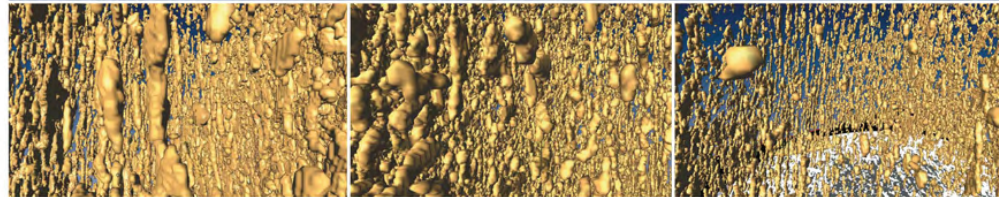
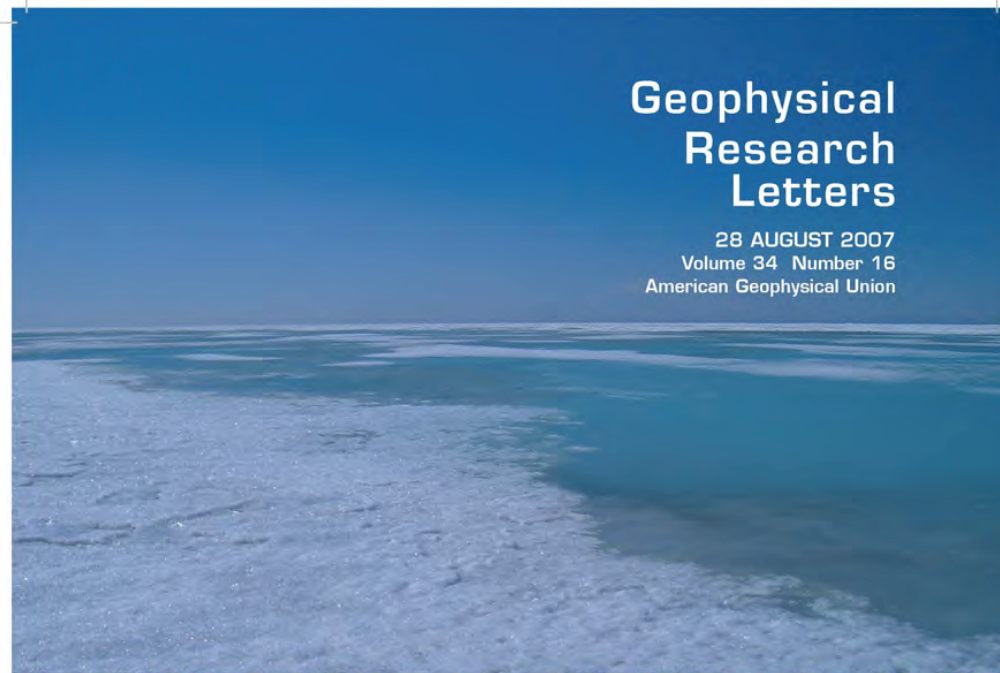
Golden, Ackley, Lytle *Science* 1998

Golden, Eicken, Heaton, Miner, Pringle, Zhu *GRL* 2007

Pringle, Miner, Eicken, Golden *J. Geophys. Res.* 2009

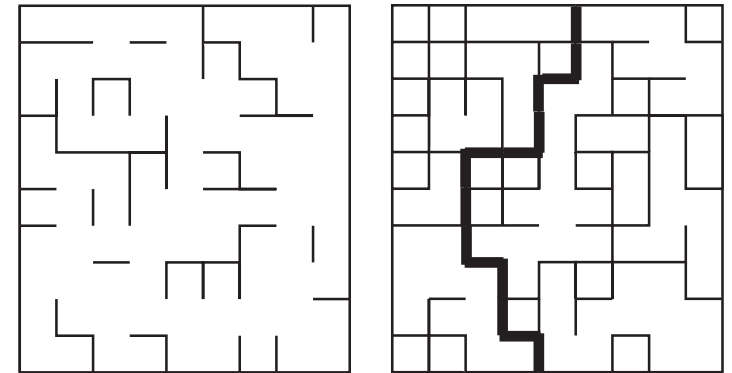
Thermal evolution of permeability and microstructure in sea ice

Golden, Eicken, Heaton, Miner, Pringle, Zhu, GRL 2007



A unified approach to understanding permeability in sea ice • Solving the mystery of booming sand dunes • Entering into the "greenhouse century": A case study from Switzerland

percolation theory for fluid permeability



percolation threshold, critical exponents

X-ray CT

confirms 'rule of fives'

Golden et al., Science 1998

Pringle, Eicken, Miner, Golden
J. Geophys. Res. 2009

theory agrees closely
with field data

macro

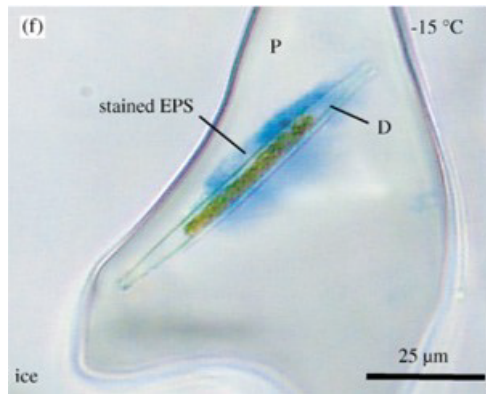


micro

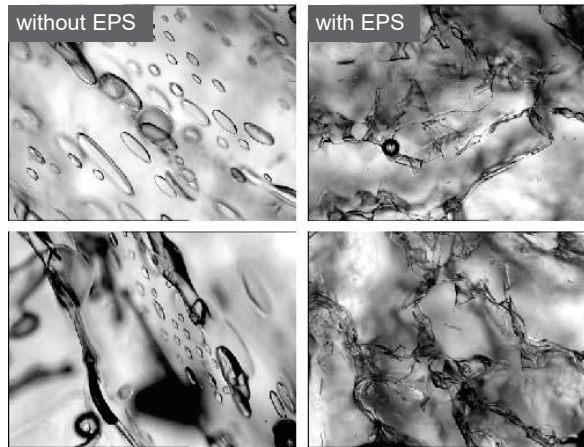
Sea ice algae secrete exopolymeric substances (EPS) affecting evolution of brine microstructure.

How does EPS affect fluid transport? How does the biology affect the physics?

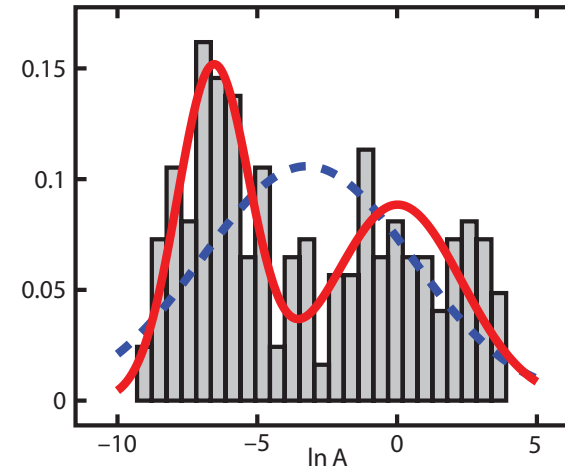
FRACTAL



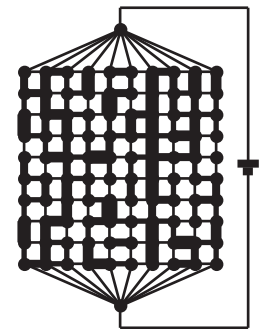
Krembs



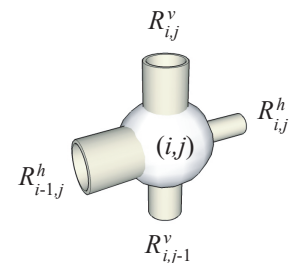
Krembs, Eicken, Deming, PNAS 2011



RANDOM PIPE MODEL



- 2D random pipe model with bimodal distribution of pipe radii
- Rigorous bound on permeability k ; results predict observed drop in k



Steffen, Epshteyn, Zhu, Bowler, Deming, Golden
Multiscale Modeling and Simulation, 2018

Zhu, Jabini, Golden,
Eicken, Morris
Ann. Glac. 2006

EPS - Algae Model Jajeh, Reimer, Golden

SIAM News
June 2024

Notices

of the American Mathematical Society

May 2009

Volume 56, Number 5

Climate Change and
the Mathematics of
Transport in Sea Ice

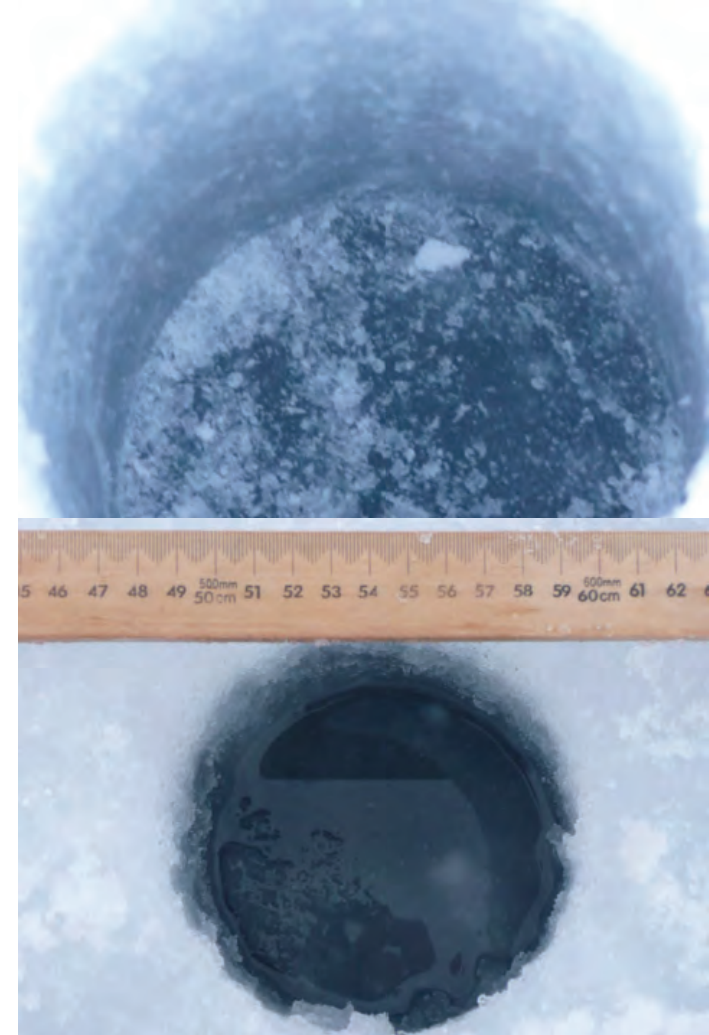
page 562

Mathematics and the
Internet: A Source of
Enormous Confusion
and Great Potential

page 586

photo by Jan Lieser

Real analysis in polar coordinates (see page 613)

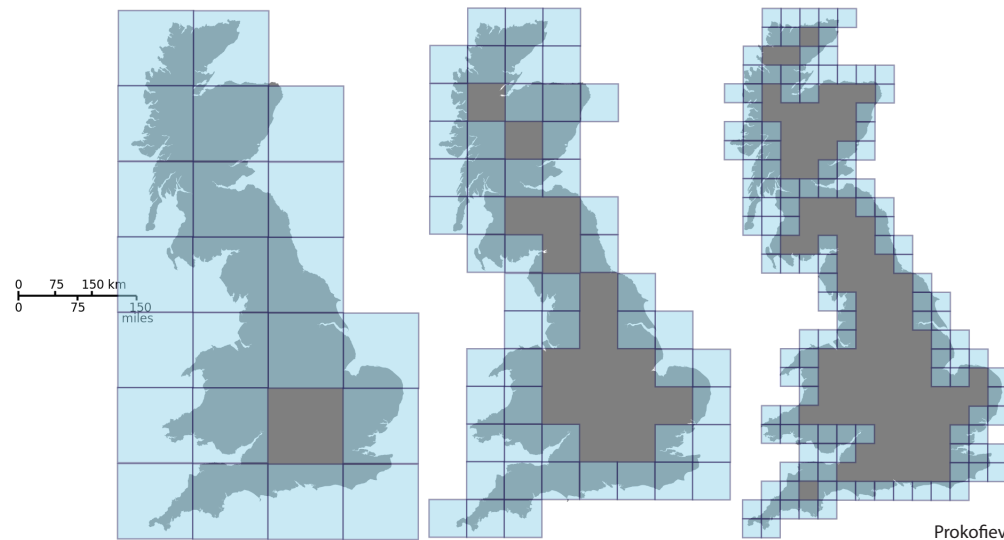


***measuring
fluid permeability
of Antarctic sea ice***

SIPEX 2007

Thermal Evolution of Brine Fractal Geometry in Sea Ice

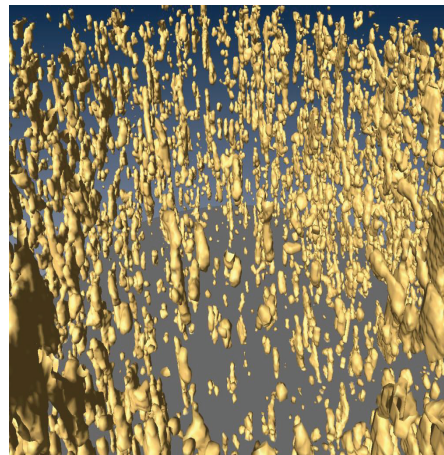
Nash Ward, Daniel Hallman, Benjamin Murphy, Jody Reimer,
Marc Oggier, Megan O'Sadnick, Elena Cherkaev and Kenneth Golden, 2025



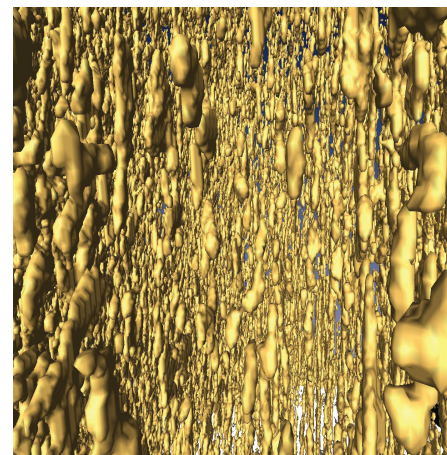
fractal dimension of the
coastline of Great Britain
by box counting

$$N(\epsilon) \sim \epsilon^{-D}$$

$T = -12^{\circ} \text{C}$, $\phi = 0.033$



$T = -8^{\circ} \text{C}$, $\phi = 0.057$



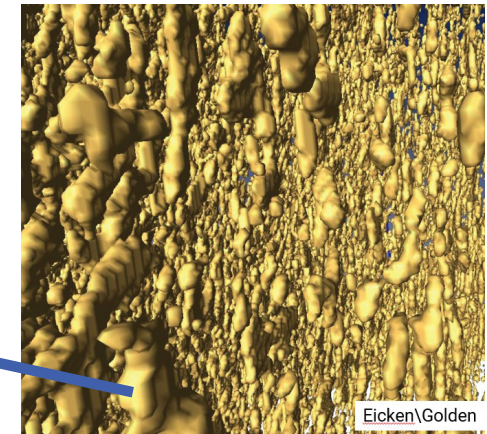
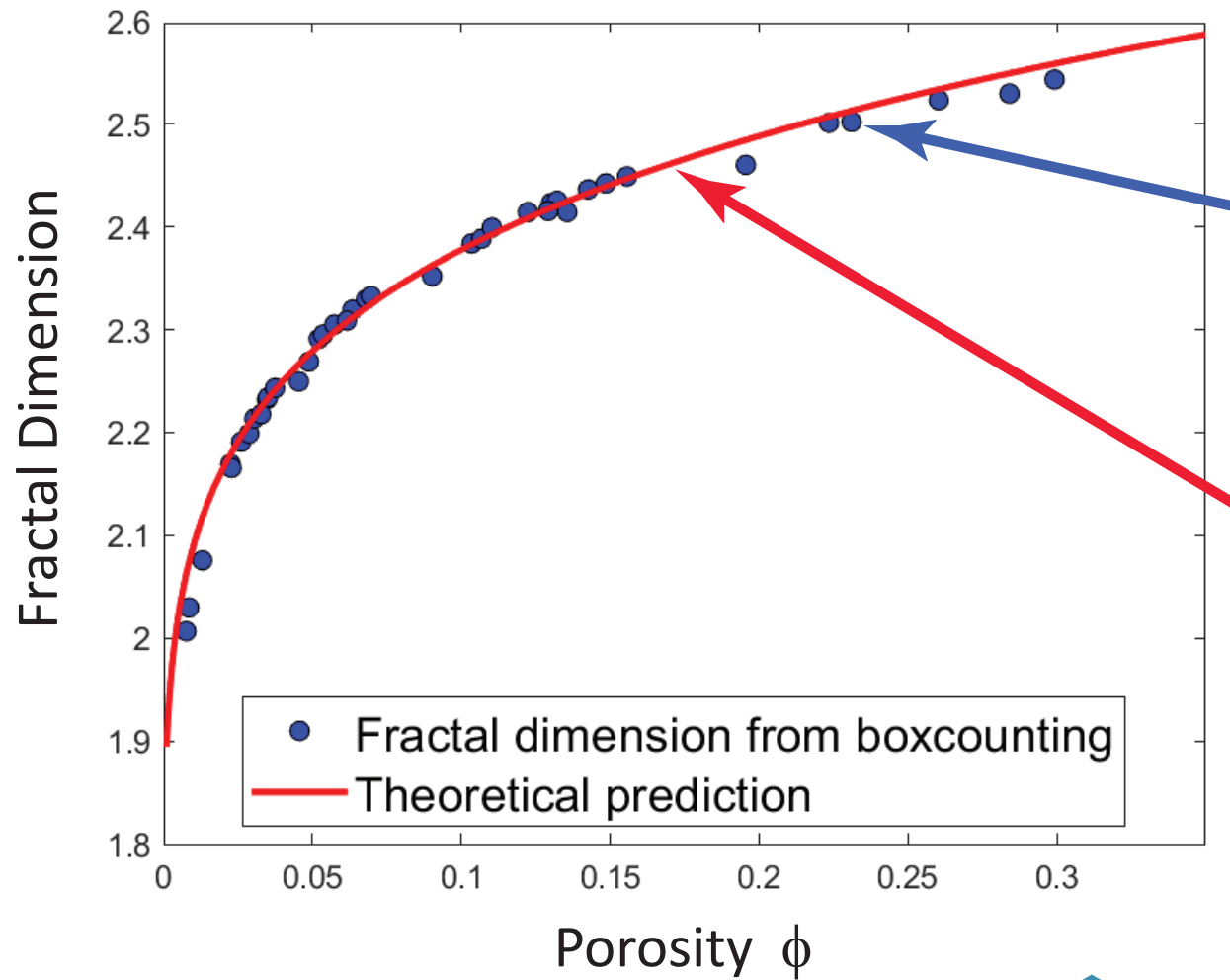
brine channels and
inclusions “look”
like fractals
(from 30 yrs ago)

X-ray computed
tomography of
brine in sea ice

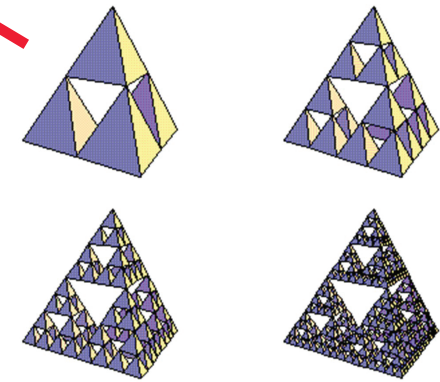
columnar and granular

Golden, Eicken, et al. *GRL*, 2007

The first quantitative study of the fractal dimension of brine in sea ice and its strong dependence on temperature and porosity.



Follows same curve as exactly self-similar Sierpinski tetrahedron



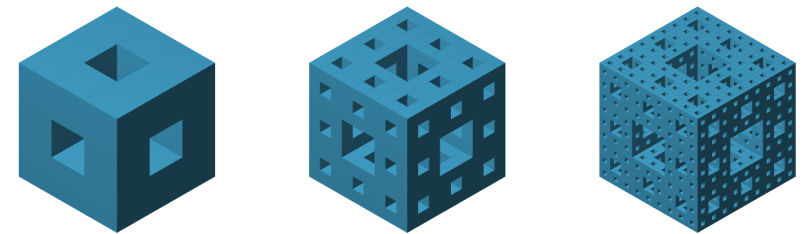
D. Eppstein

red curve

$$F_d = d_E - \frac{\ln \phi}{\ln(\lambda_{min}/\lambda_{max})}$$

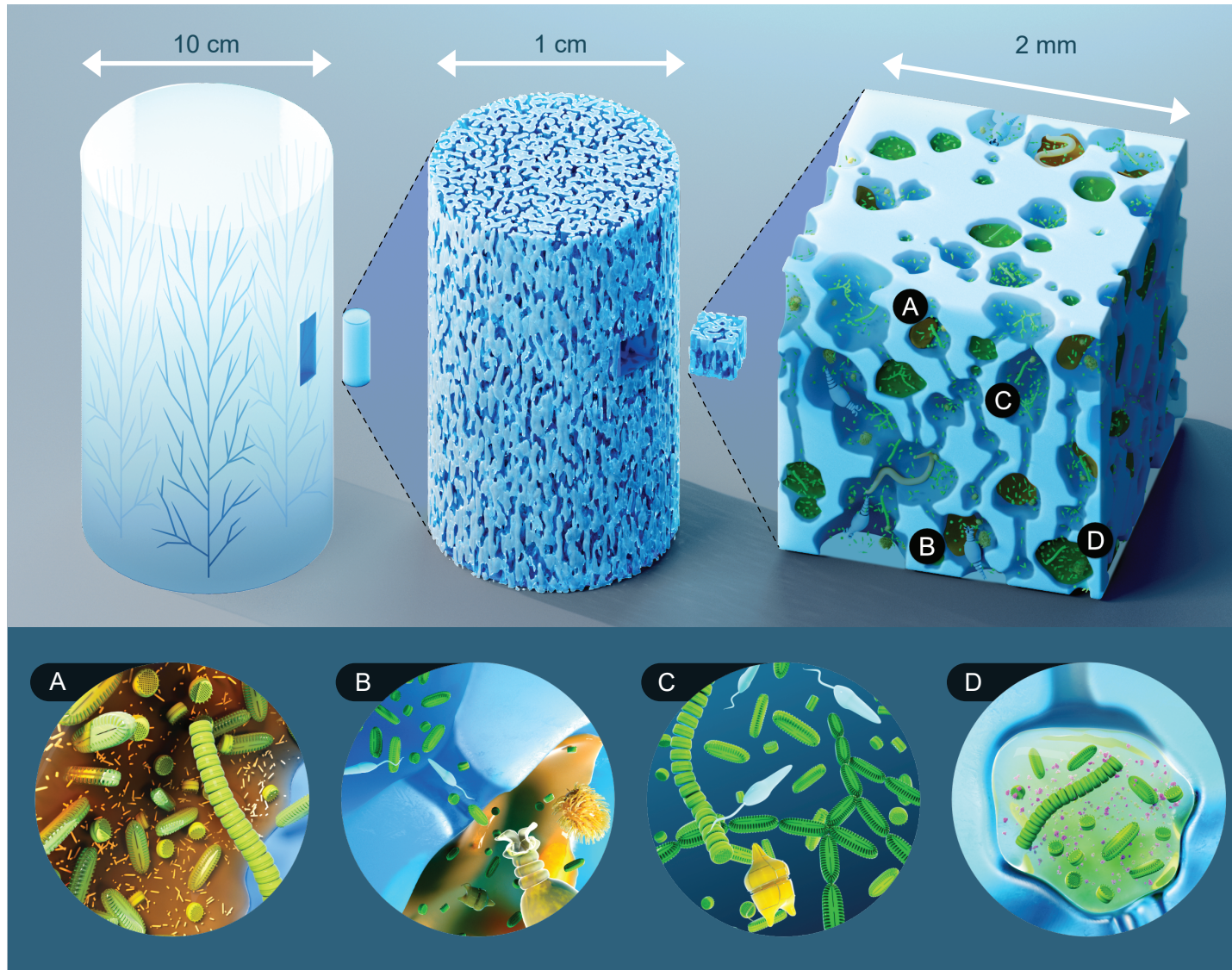
Katz and Thompson, 1985; Yu and Li, 2001

discovered for sandstones
statistically self-similar porous media



Fractal geometry of brine in sea ice, Ward, *et al.* 2025

Implications of brine fractal geometry on sea ice ecology and biogeochemistry



Brine inclusions are home to ice endemic organisms, e.g., bacteria, diatoms, flagellates, rotifers, nematodes.

The habitability of sea ice for these organisms is inextricably linked to its complex brine geometry.

- (A) Many sea ice organisms attach themselves to inclusion walls; inclusions with a higher fractal dimension have greater surface area for colonization.
- (B) Narrow channels prevent the passage of larger organisms, leading to refuges where smaller organisms can multiply without being grazed, as in (C).
- (D) Ice algae secrete extracellular polymeric substances (EPS) which alter inclusion geometry and may further increase the fractal dimension.

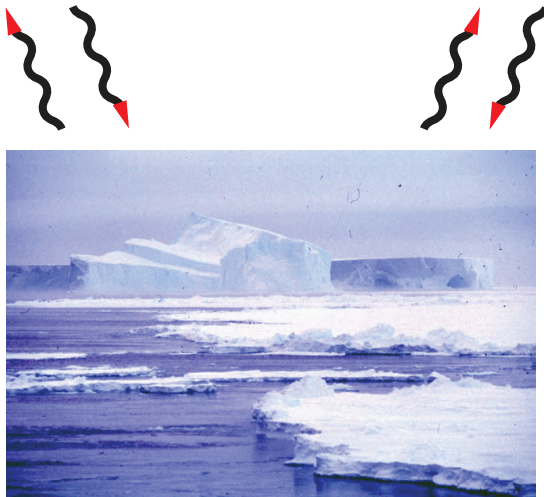


Remote Sensing of Sea Ice

with radar, microwaves, ...



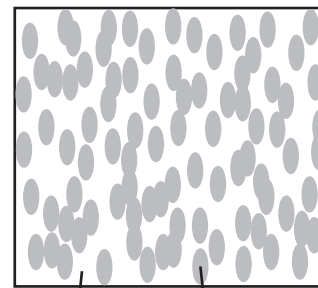
interaction of
EM waves with
brine and
polycrystalline
microstructures,
rough surfaces



INVERSE PROBLEM

Recover sea ice properties from
electromagnetic (EM) data ϵ^*

Effective complex permittivity of a composite
in the quasistatic (long wavelength) limit



ϵ_1 ϵ_2

ϵ^*

p_1, p_2 = volume fractions of
the components

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

electrical conductivity
thermal conductivity
magnetic permeability
diffusivity

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2}, \text{ composite geometry} \right)$$

**What are the effective propagation characteristics
of an EM wave (radar, microwaves) in the medium?**

Analytic Continuation Method for Homogenization

Bergman 1978, Milton 1979, Golden & Papanicolaou 1983, Milton 2002

Stieltjes integrals for homogenized parameters separate component parameters from geometry

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z} \quad s = \frac{1}{1 - \epsilon_1/\epsilon_2}$$

← geometry
← material parameters

- spectral measure of self adjoint operator $\Gamma\chi$ (matrix)

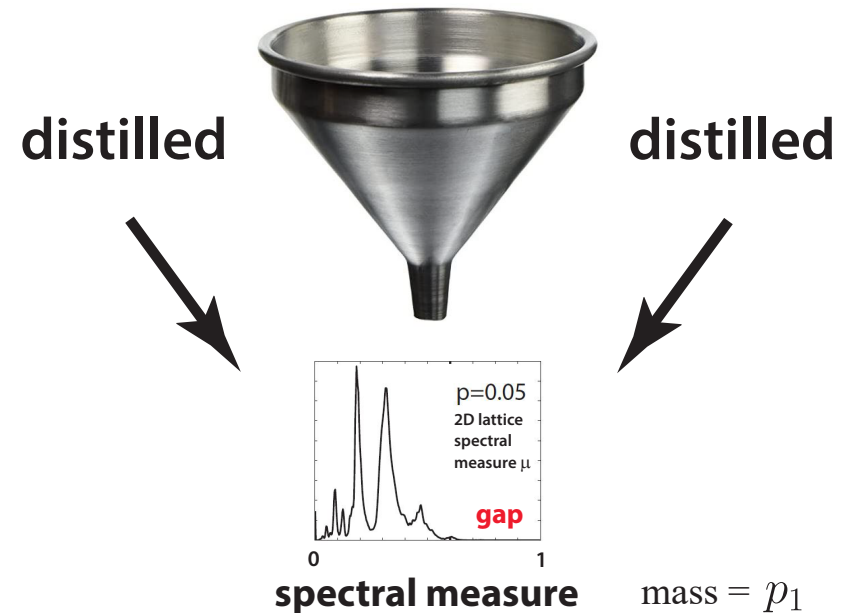
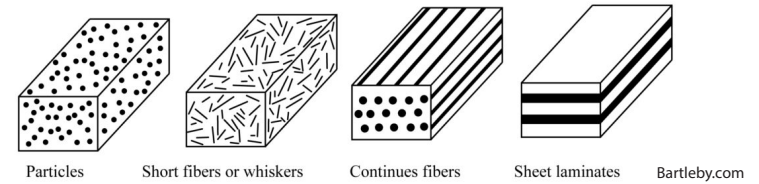
$$\Gamma = \nabla(-\Delta)^{-1}\nabla.$$

χ = characteristic function of the brine phase

- $E = s (s + \Gamma\chi)^{-1} e_k$ **resolvent**

- bounds in the complex plane; approximations
inverse bounds to recover porosity, connectivity

complexities of mixture geometry



spectral properties of operator (matrix)
~ quantum states, energy levels for atoms

eigenvectors

eigenvalues

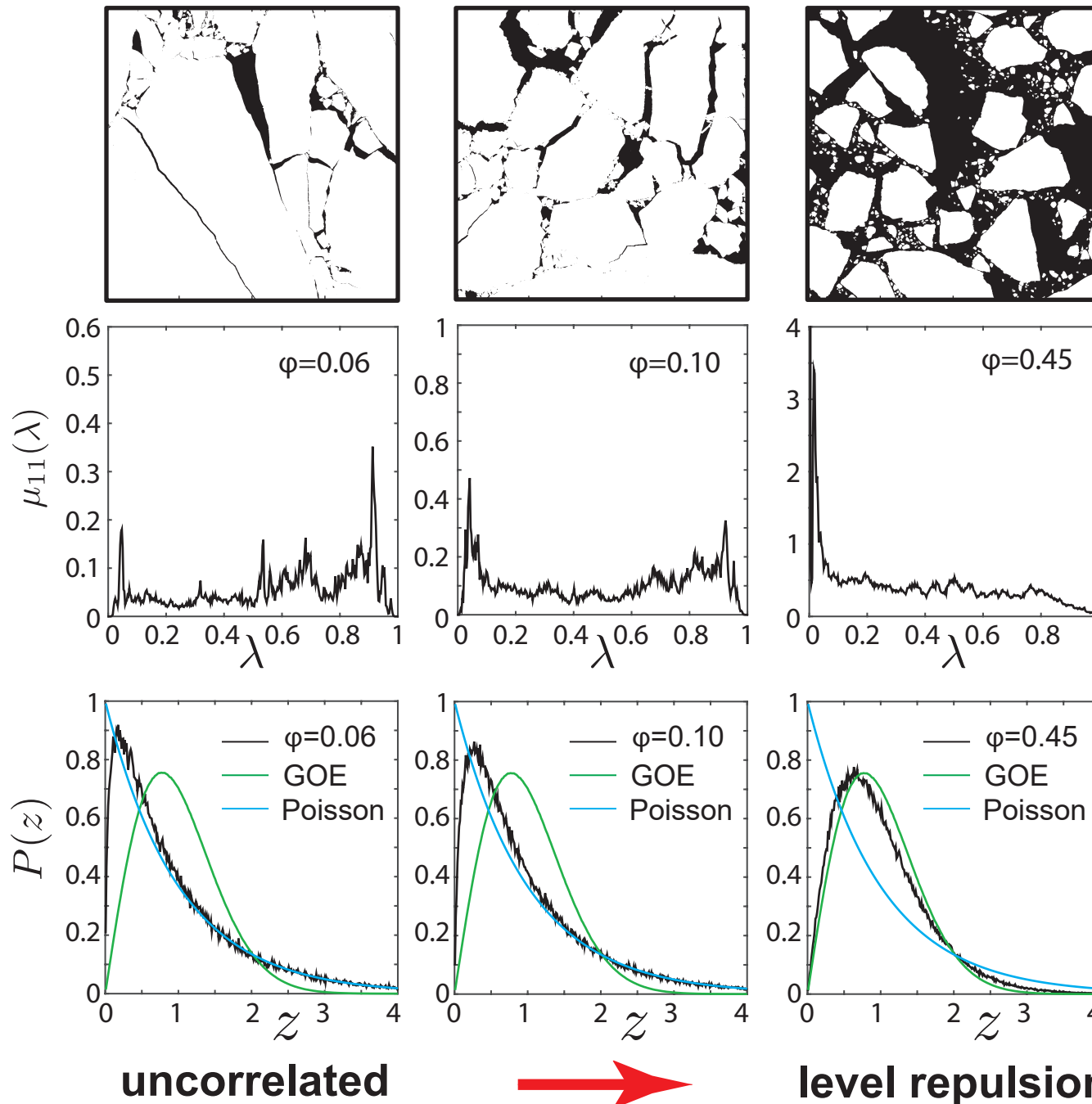
$\Gamma\chi$: microscale → macroscale

Spectral computations for sea ice floe configurations

spectral
measures

eigenvalue
spacing
distributions

**RANDOM
MATRIX
THEORY**



**UNIVERSAL
Wigner-Dyson
distribution**

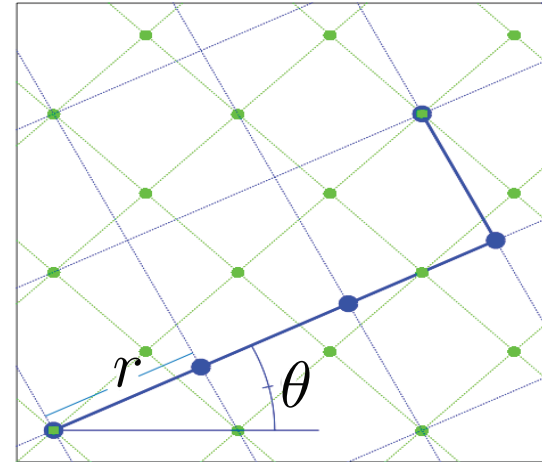
**Anderson
localization
transitions**

**zeros
Riemann
zeta fn.**

Moiré patterns generate two component composites on any scale

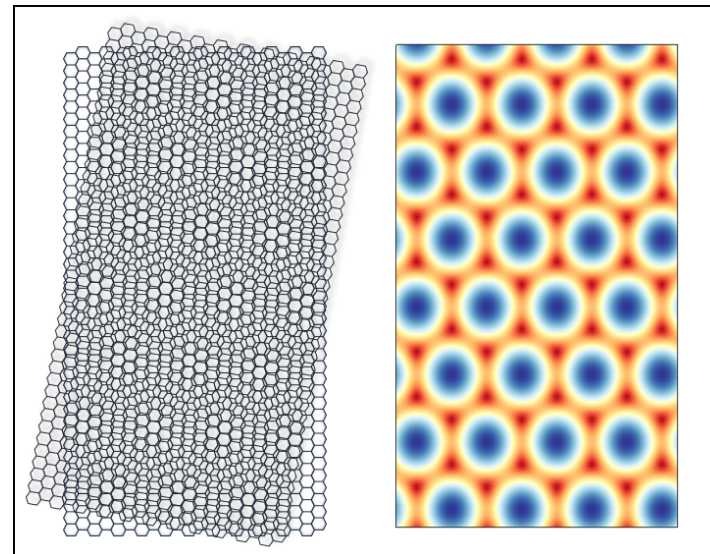
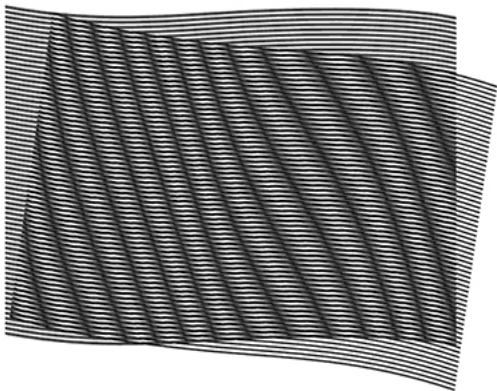
rotation
dilation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = r \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



$$\psi(x', y') = \cos 2\pi x' \cos 2\pi y'$$

$$\chi = \begin{cases} 1, & \psi \geq 0 \\ 0, & \psi < 0 \end{cases}$$



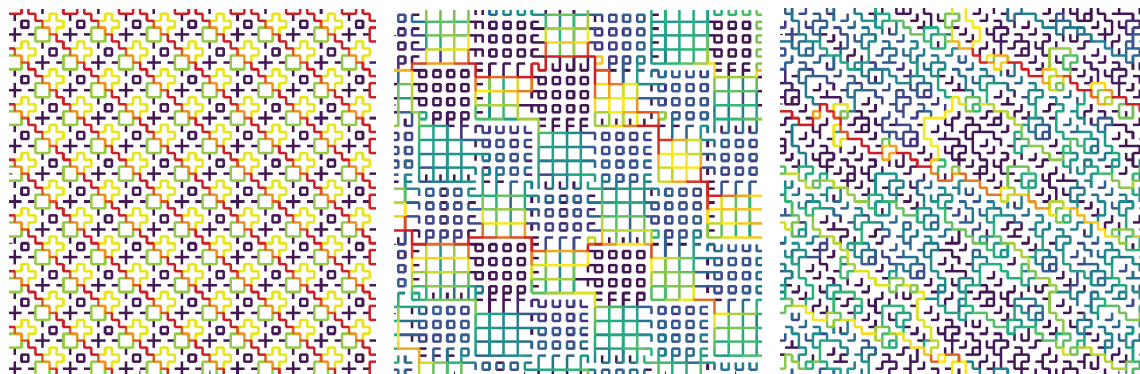
quantum dots
artificial atoms

Tran et al.
Nature 2019

twisted bilayer composites

Morison, Murphy, Cherkhev,
Golden, Comm. Phys. 2022

Moiré Pattern Creates Tunable Microgeometry



**Moiré composites
with exotic properties**

can be engineered on any scale!

inspired by sea ice studies

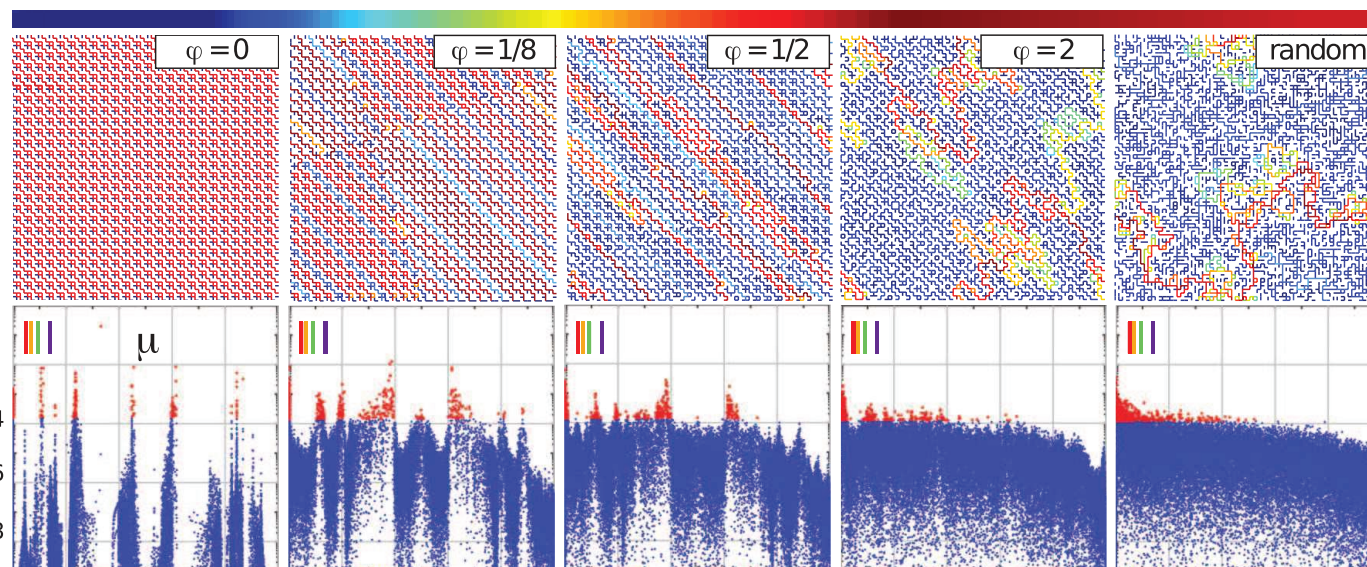
periodic



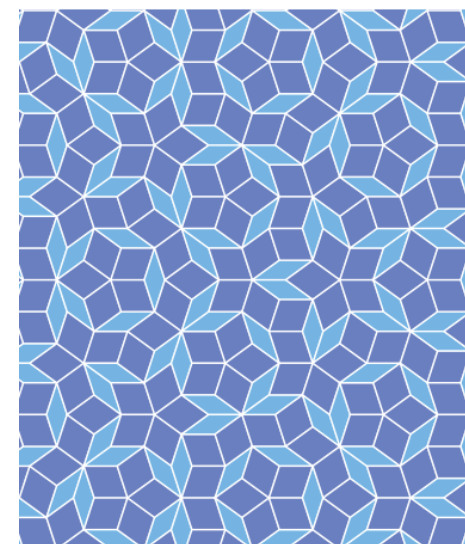
quasiperiodic

quasicrystals

electric field
strength



Anderson transition as twist angle is tuned



aperiodic tiling - R. Penrose

**solid state physics of electronic
transport in semiconductors**



**classical transport in
two phase composites**

Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**
Milton (1981, 2002), Barabash and Stroud (1999), ...
- **Forward and inverse bounds**
orientation statistics
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



PROCEEDINGS A

350 YEARS
OF SCIENTIFIC
PUBLISHING

An invited review
commemorating 350 years
of scientific publishing at the
Royal Society

A method to distinguish
between different types
of sea ice using remote
sensing techniques

A computer model to
determine how a human
should walk so as to expend
the least energy



THE
ROYAL
SOCIETY
PUBLISHING

higher threshold for fluid flow in granular sea ice

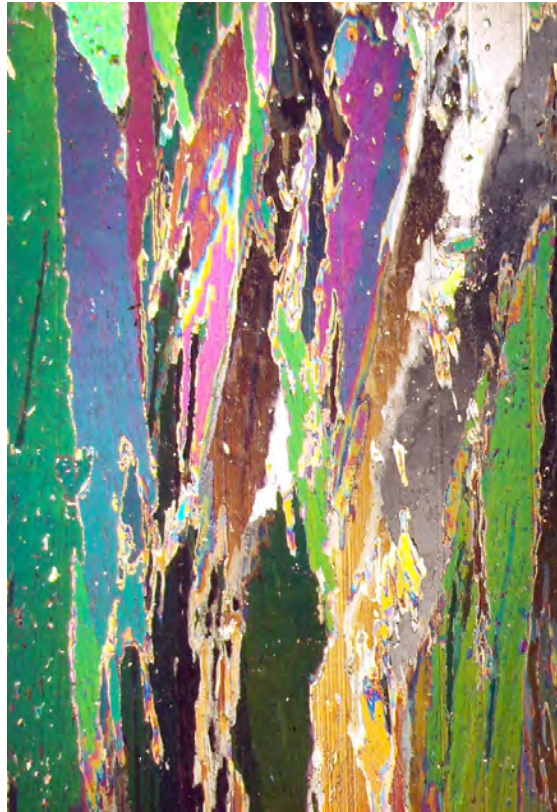
microscale details impact “mesoscale” processes

nutrient fluxes for microbes
melt pond drainage
snow-ice formation

columnar

granular

5%



10%



Golden, Furse, Gully, Lin, Mosier, Sampson, Tison 2025

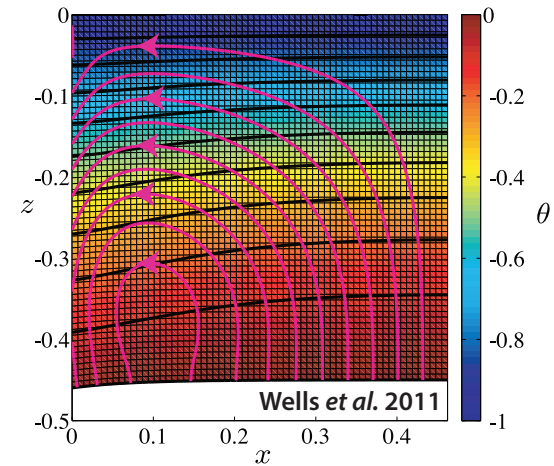
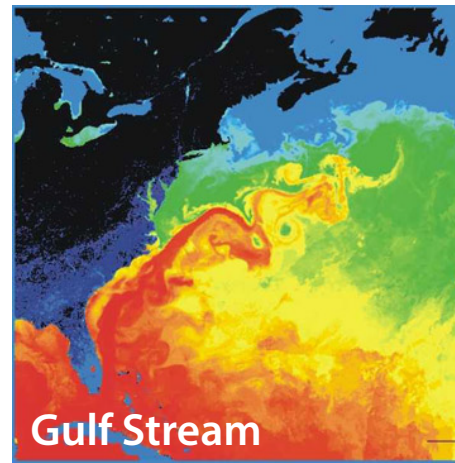
electromagnetically distinguish ice types
inverse homogenization for polycrystals

mesoscale

advection enhanced diffusion

HOMOGENIZE

heat transport in sea ice with convection
nutrient and salt transport in sea ice
sea ice floes in winds and ocean currents
tracers, buoys diffusing in ocean eddies



advection diffusion
equation with
velocity field $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

homogenize

heat equation

$$\frac{\partial \bar{T}}{\partial t} = \kappa^* \Delta \bar{T}$$

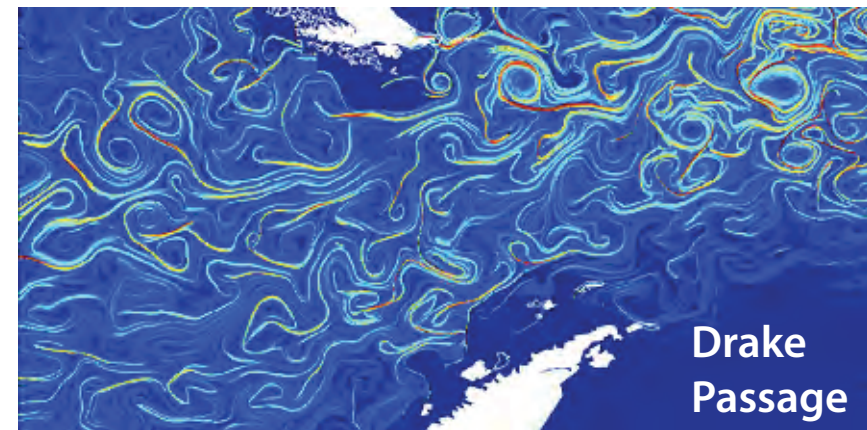
κ^* **effective diffusivity**

Stieltjes integral for κ^ with spectral measure*

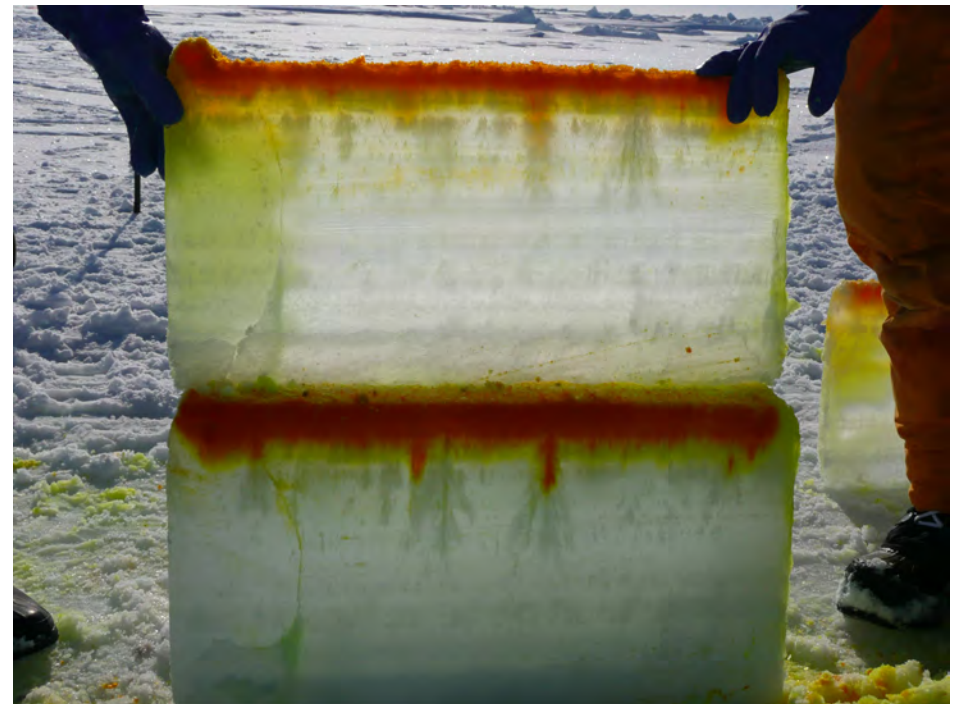
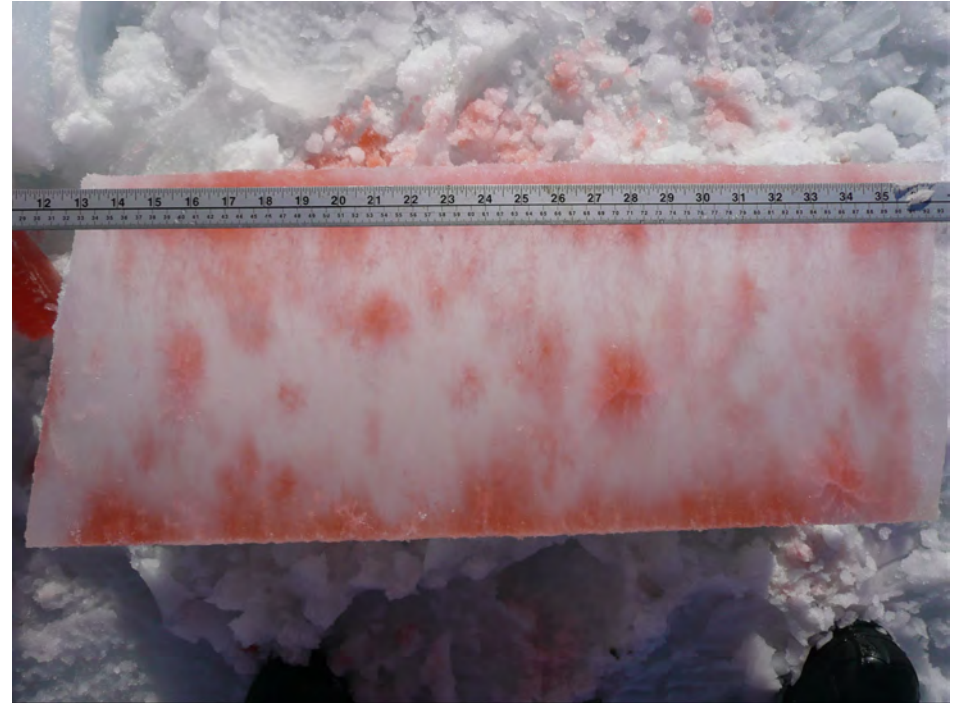
Avellaneda and Majda, PRL 89, CMP 91

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017

Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020

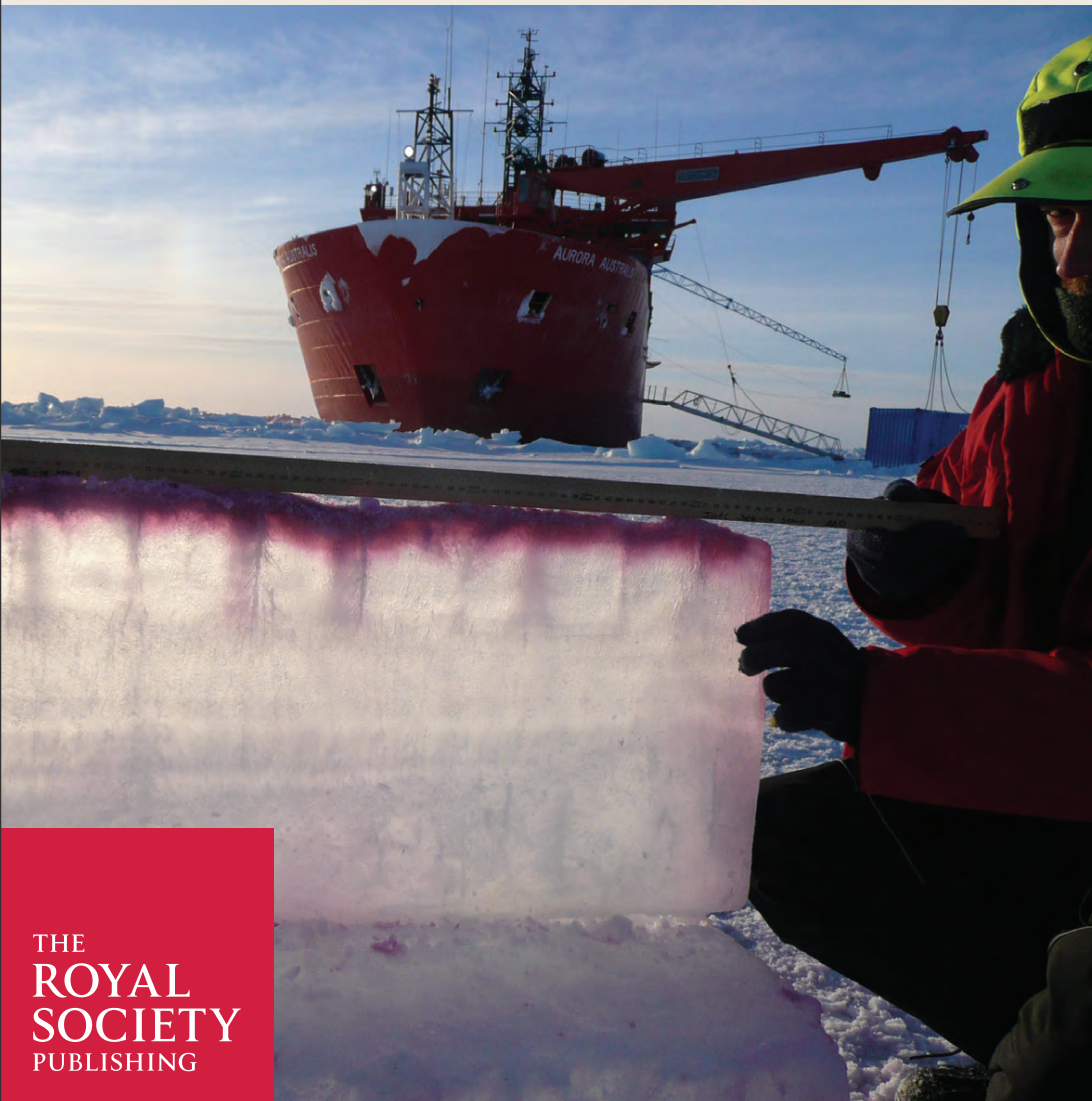


tracers flowing through inverted sea ice blocks



PROCEEDINGS OF THE ROYAL SOCIETY A

MATHEMATICAL, PHYSICAL AND ENGINEERING SCIENCES



Homogenization for convection-enhanced thermal transport in sea ice

N. Kraitzman, R. Hardenbrook,
H. Dinh, N. B. Murphy, E. Cherkaev,
J. Zhu and K. M. Golden

August 2024

First rigorous mathematical theory of
thermal conductivity of sea ice with
convective fluid flow; captures data.

missing in climate models

ocean wave propagation through the sea ice pack



- wave-ice interactions critical to growth and melting processes
- break-up; pancake promotion floe size distribution

HOMOGENIZE
ice-ocean composite

find effective
complex viscoelasticity

[Anderson localization]

Keller 1998

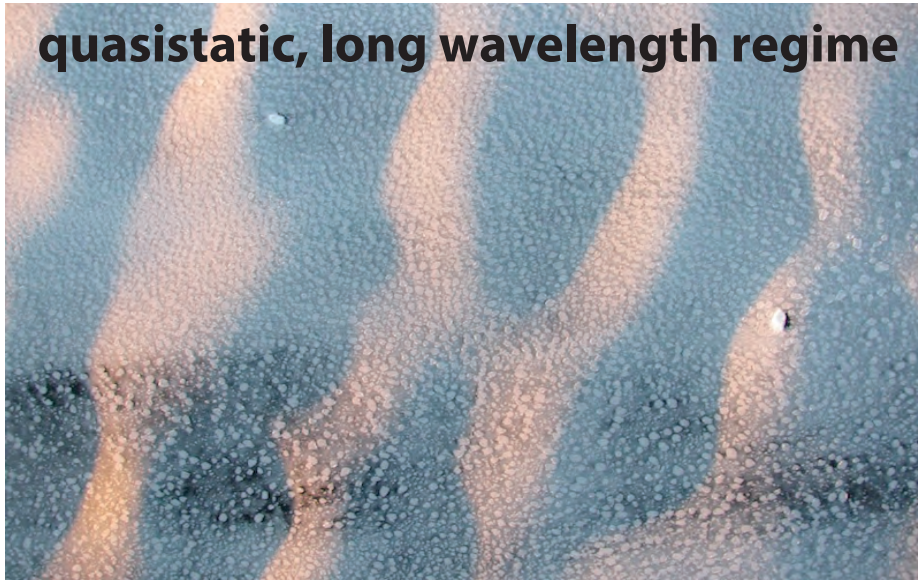
Mosig, Montiel, Squire 2015

Wang, Shen 2012

quasistatic, long wavelength regime

homogenized
parameter
depends on
sea ice
concentration
and ice floe
geometry

like
EM waves

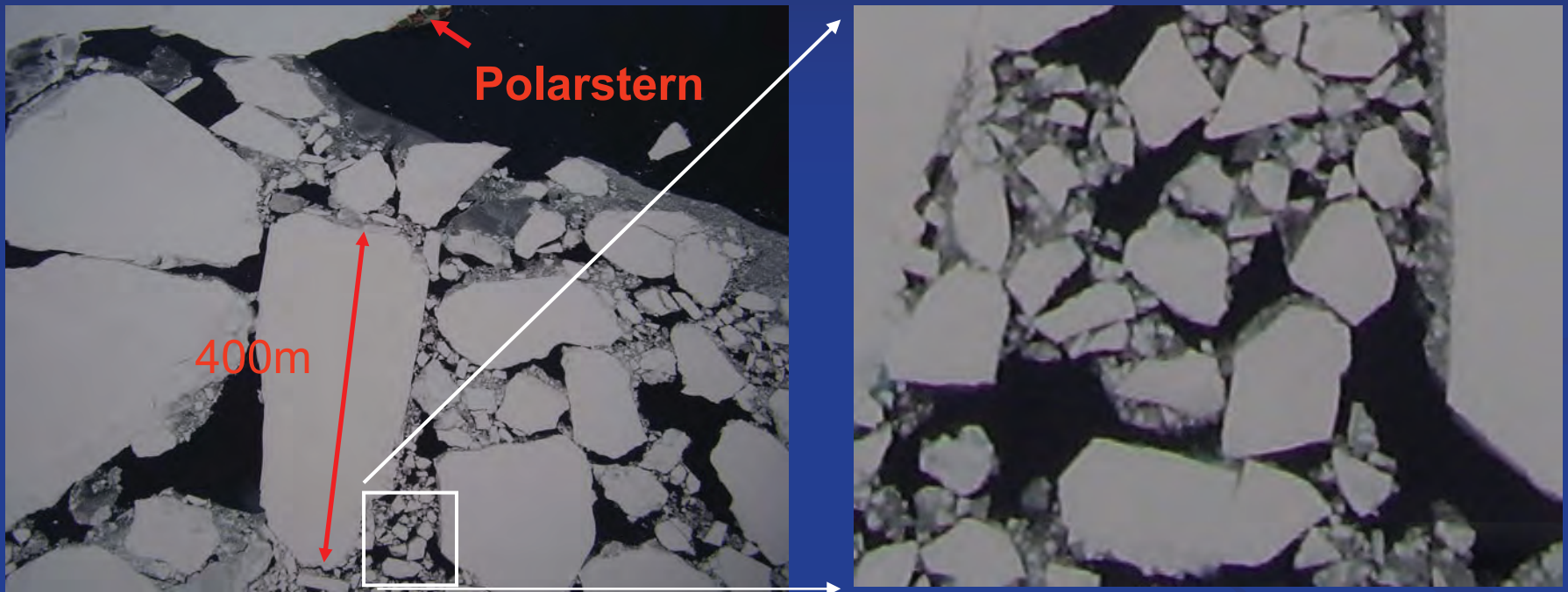


Sampson, Murphy, Hallman, Cherkaev, Golden 2025

The sea ice pack has fractal structure.

Self-similarity of sea ice floes

Weddell Sea, Antarctica



***fractal dimensions of Okhotsk Sea ice pack
smaller scales $D \sim 1.2$, larger scales $D \sim 1.9$***

fractal dim. vs. floe size exponent

Adam Dorsky, Nash Ward, Ken Golden 2025

Toyota, et al. *Geophys. Res. Lett.* 2006

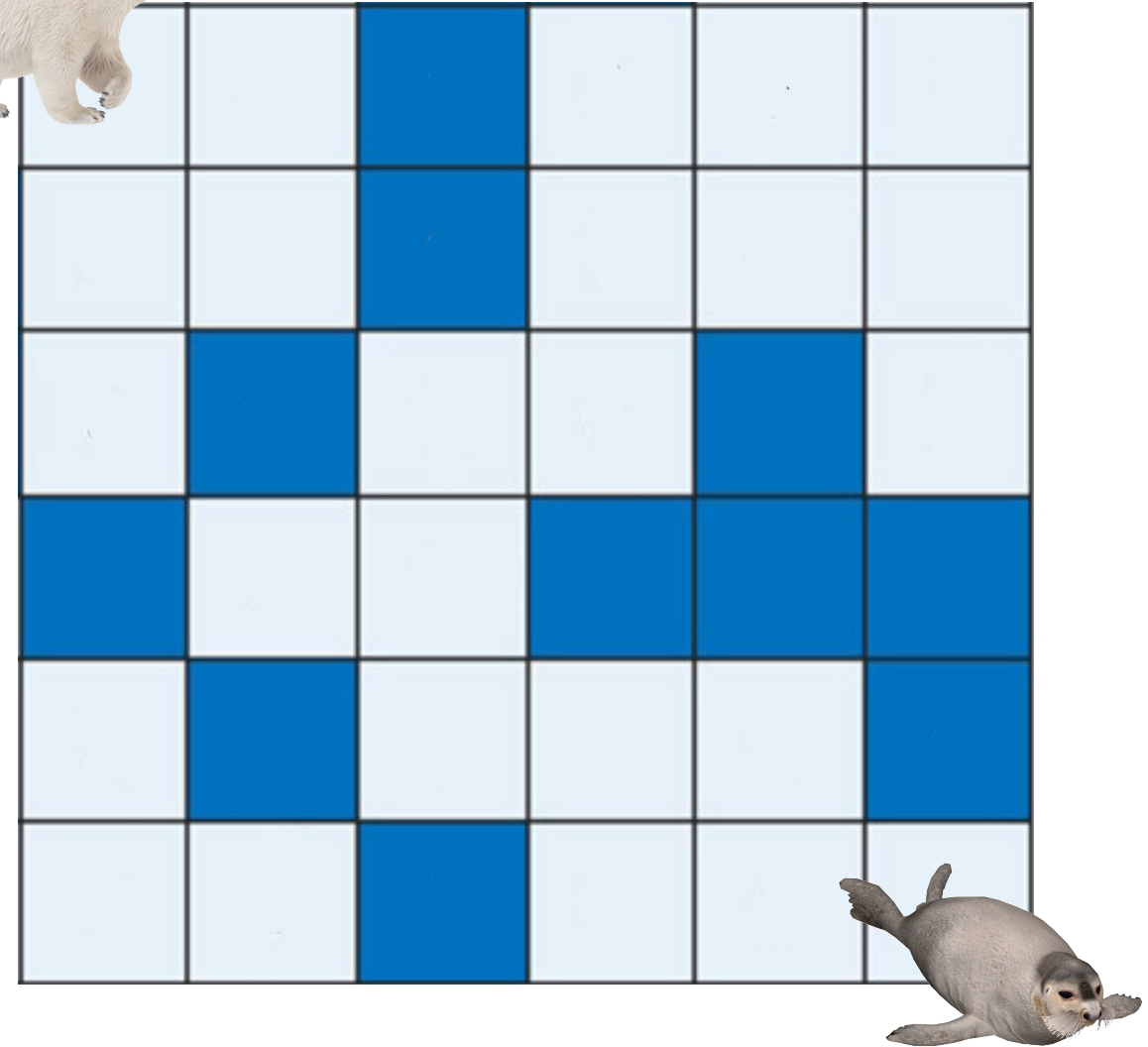
Rothrock and Thorndike, *J. Geophys. Res.* 1984

Optimal Movement of a Polar Bear in a Heterogenous Icescape

Nicole Forrester, Jody Reimer, Ken Golden 2024

Polar bears expend 5X more energy swimming than walking on sea ice.

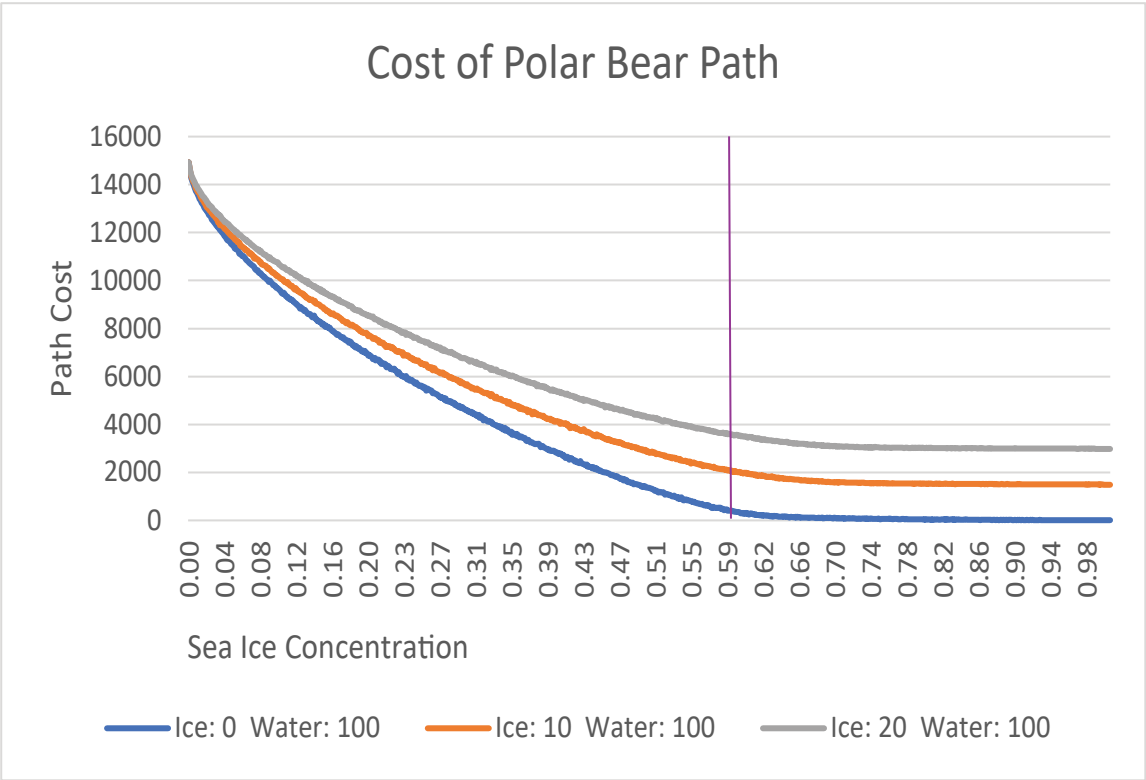
As sea ice is lost, how do polar bears optimize their movement to save energy and survive?



Polar Bear Percolation

To study the importance of ice connectedness, we exaggerate the data by setting the cost of walking on ice to 0 with the cost of swimming still at 5.

$C(p)$



$$h = \frac{C_i}{C_w}$$

ratio of local
“conductivities”

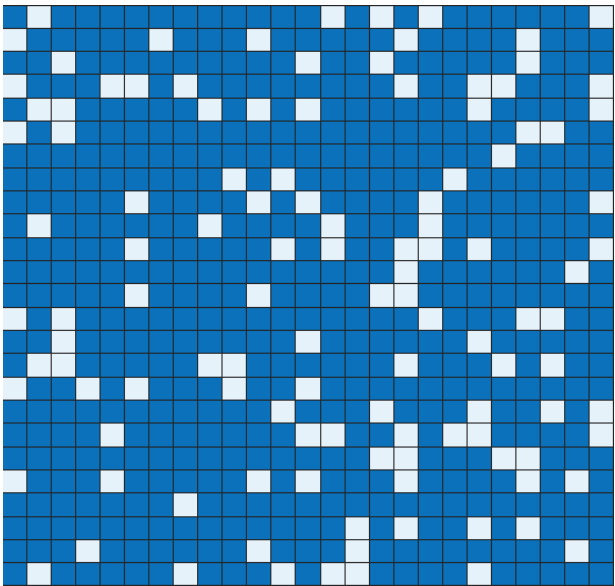
← $h = 0.2$
← $h = 0.1$
← $h = 0$

site percolation
threshold

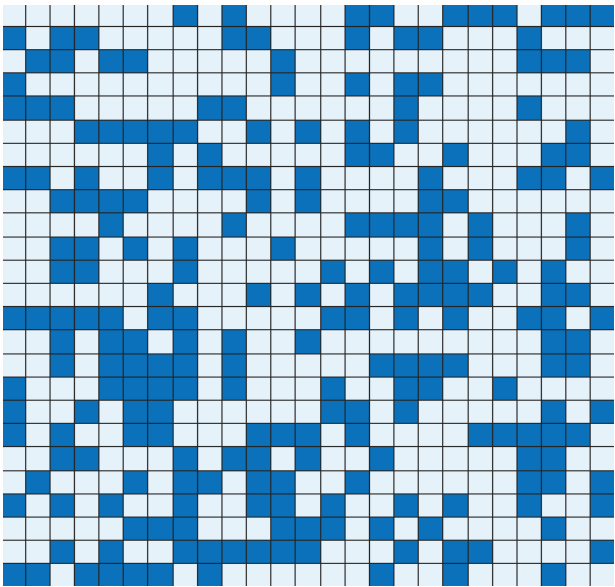
$p_c = 0.59$ for $d = 2$

Polar Bear
Critical
Exponent

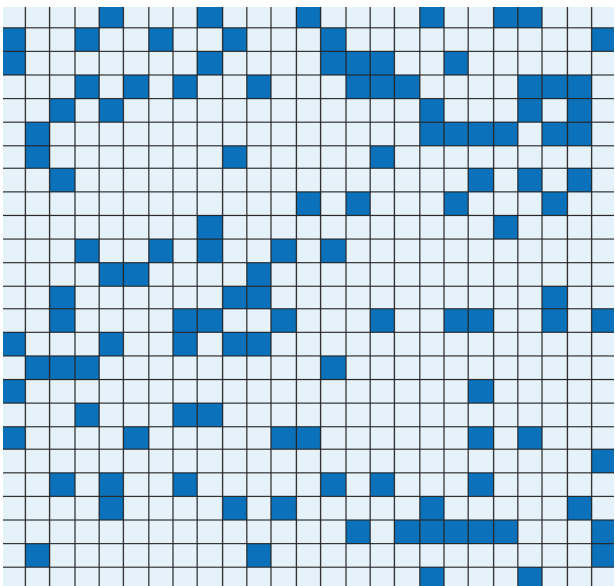
← $h = 0$



20% Ice

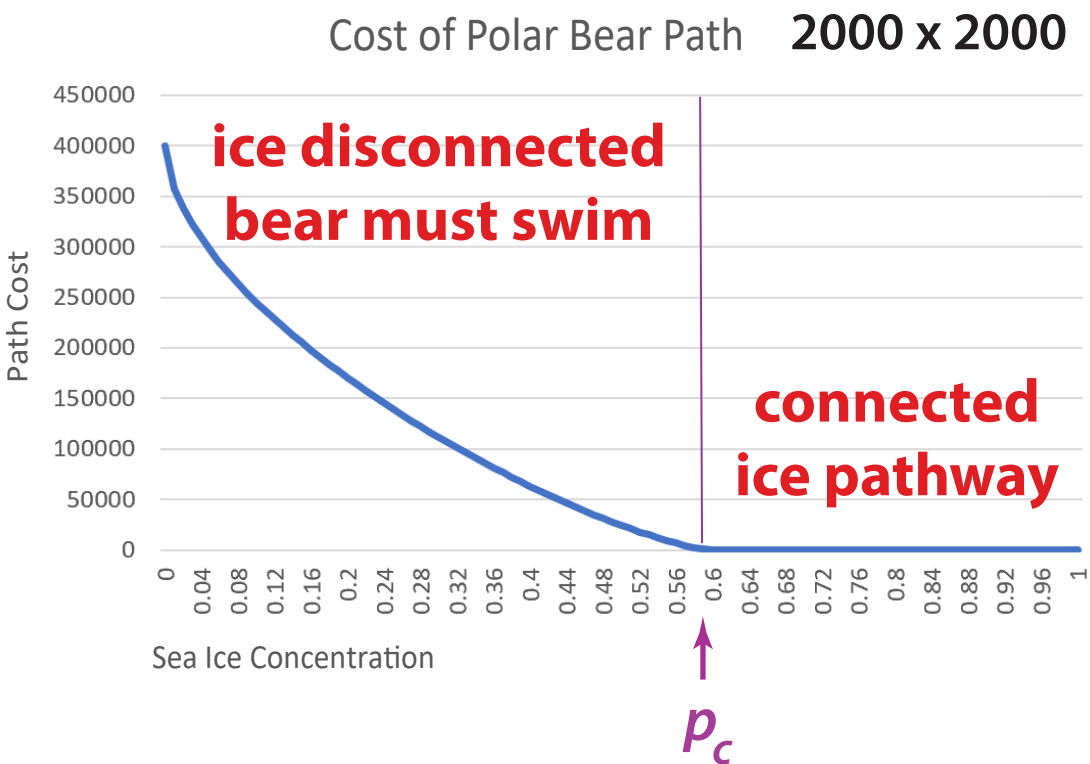


60% Ice



80% Ice

$C(p)$

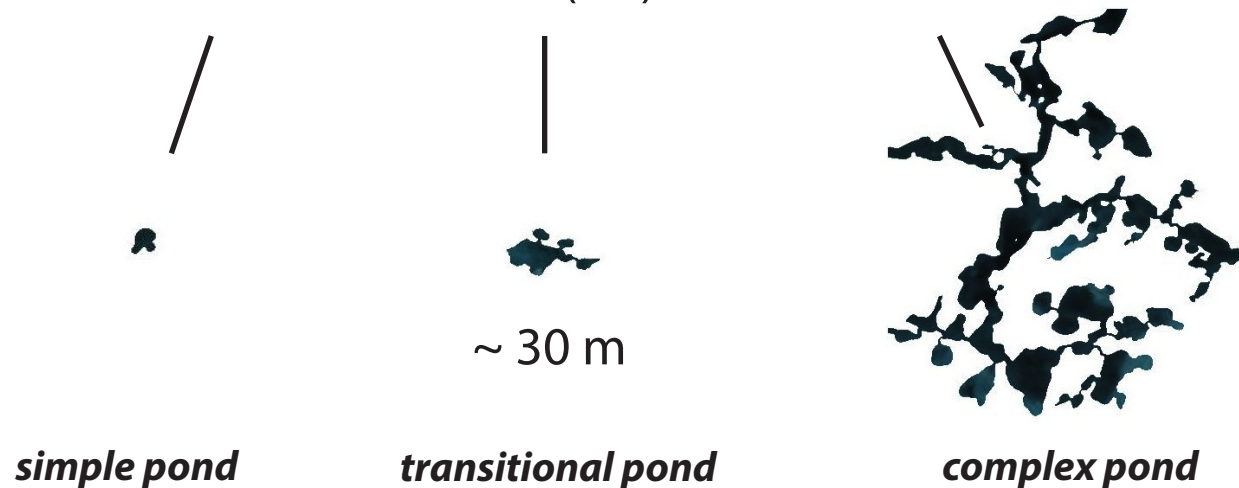
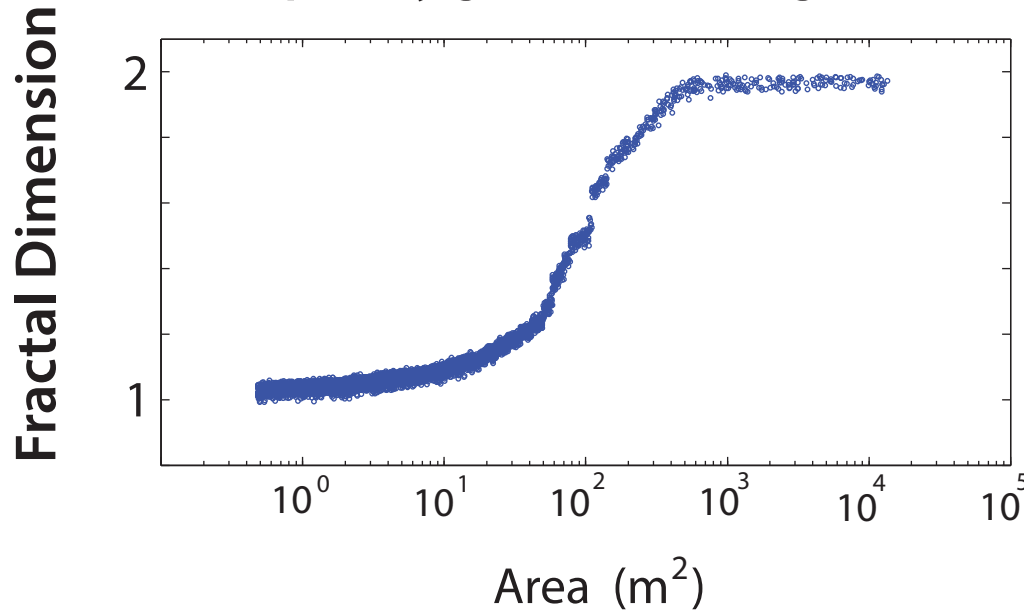


Transition in the fractal geometry of Arctic melt ponds

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

The Cryosphere, 2012

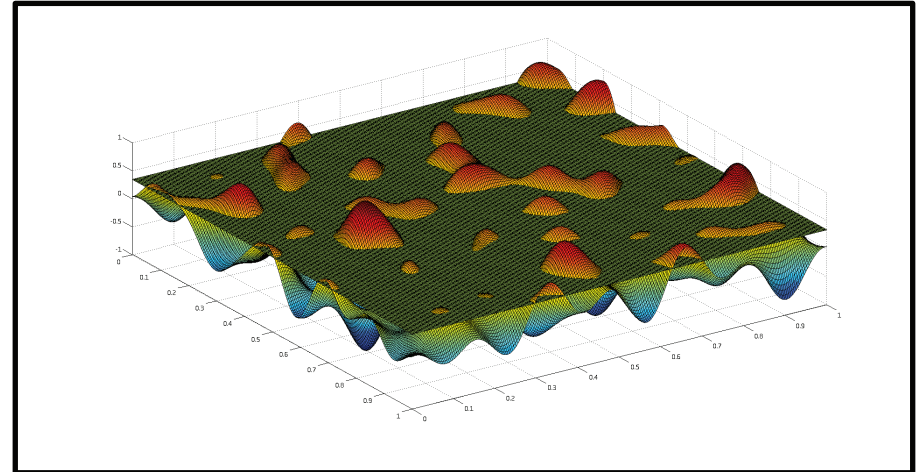
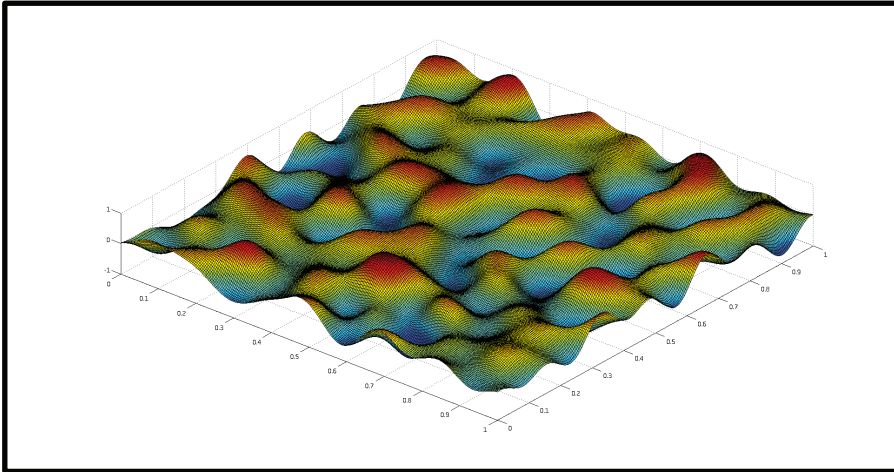
complexity grows with length scale



Continuum percolation model for melt pond evolution

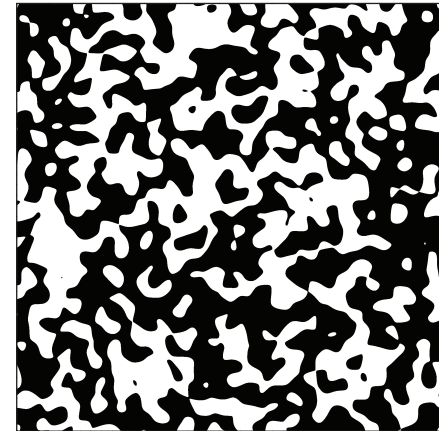
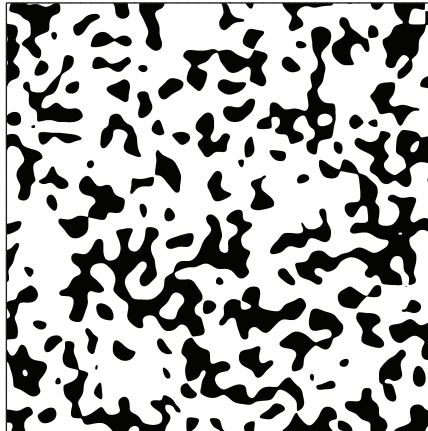
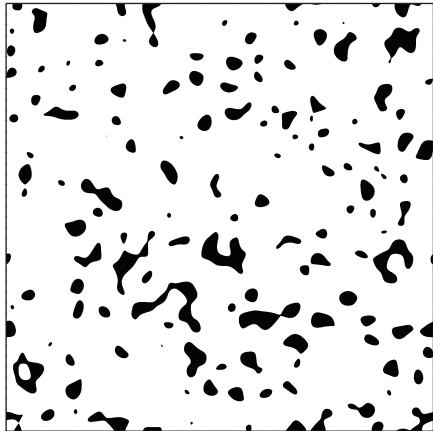
level sets of random surfaces

Brady Bowen, Court Strong, Ken Golden, J. Fractal Geometry 2018



random Fourier series representation of surface topography

intersections of a plane with the surface define melt ponds



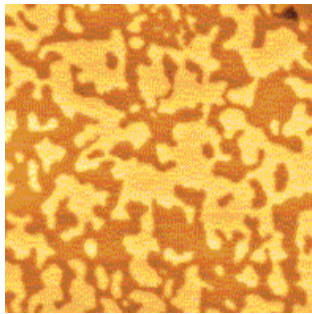
electronic transport in disordered media

diffusion in turbulent plasmas

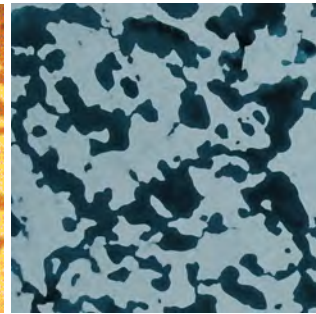
Isichenko, Rev. Mod. Phys., 1992

From magnets to melt ponds

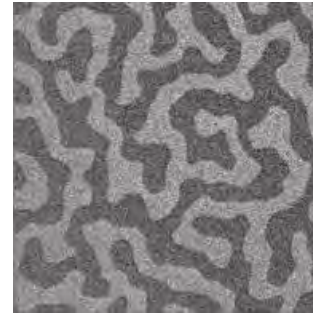
100 year old model for magnetic materials
used to explain melt pond geometry



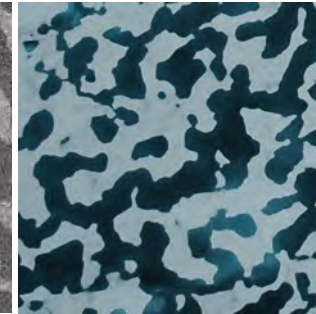
magnetic domains
in cobalt



Arctic melt ponds

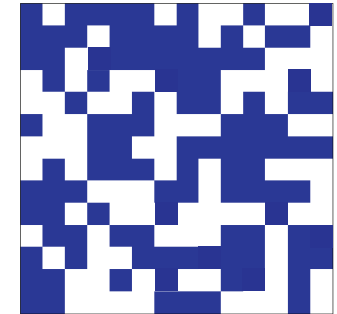
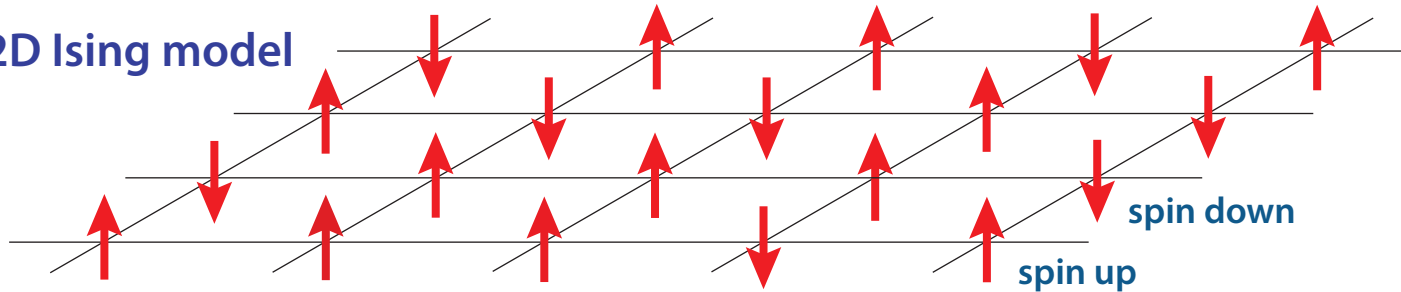


magnetic domains
in cobalt-iron-boron

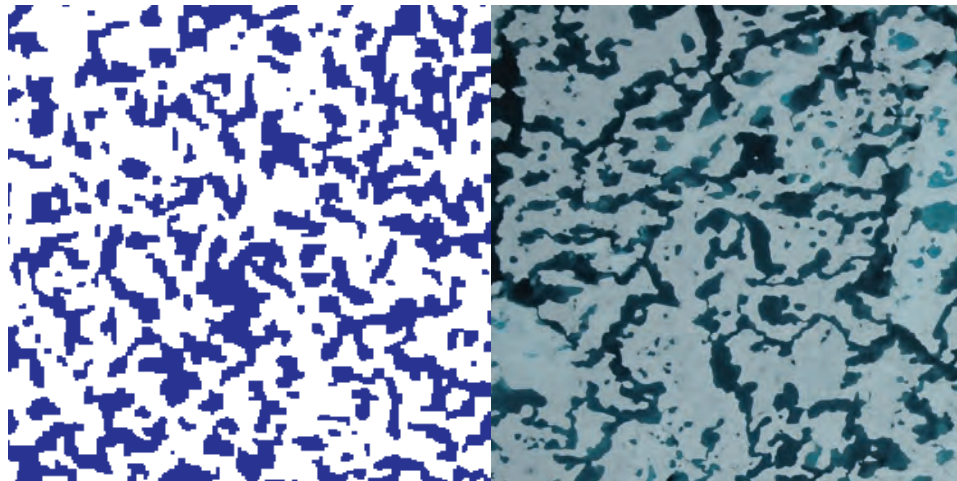


Arctic melt ponds

2D Ising model



model



real ponds
(Perovich)

Ma, Sudakov, Strong,
Golden, *New J. Phys.* 2019

Scientific American,
EOS, PhysicsWorld, ...

macroscale

Marginal Ice Zone (MIZ)

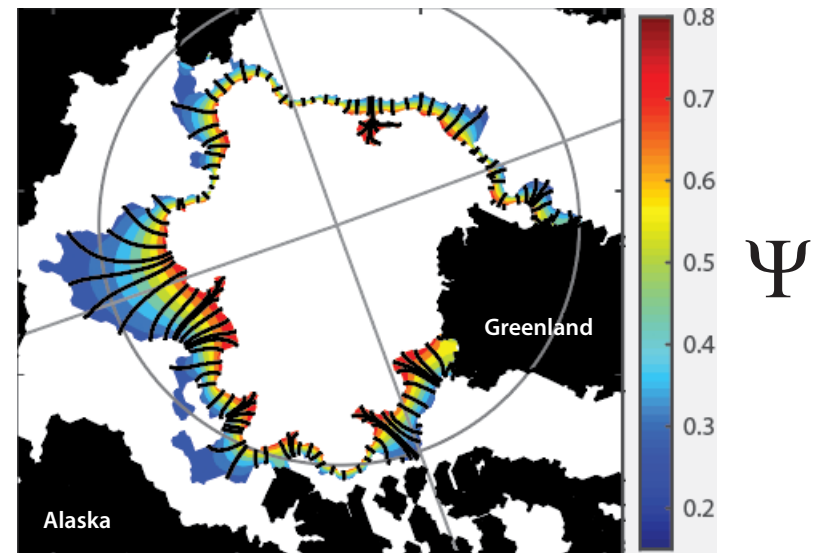
transitional region between
dense interior pack $\Psi > 0.8$
sparse outer fringes $\Psi < 0.15$

- biologically active region
- intense wave-ice interactions
- strong air-ice-ocean exchanges



MIZ WIDTH

fundamental length scale of
ecological and climate dynamics



streamlines of harmonic Ψ

MIZ width definition from medical imaging

Strong, *Climate Dynamics* 2012
Strong & Rigor, *GRL* 2013

**39% widening
1979 - 2012**

Strong, Foster, Cherkaev, Eisenman, Golden, *J. Atmos. Oceanic Tech.* 2017

Strong & Golden, *SIAM News* 2017

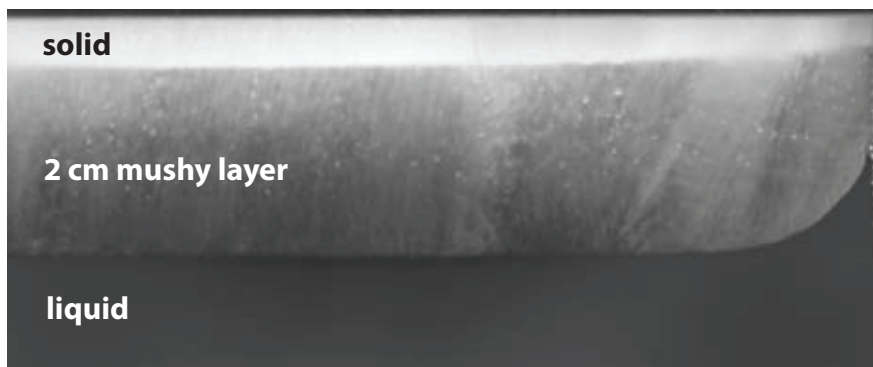
Multiscale mushy layer model for marginal ice zone dynamics

Strong, Cherkaev, Golden *Scientific Reports* 2024

MIZ - transitional region between dense pack ice and open ocean

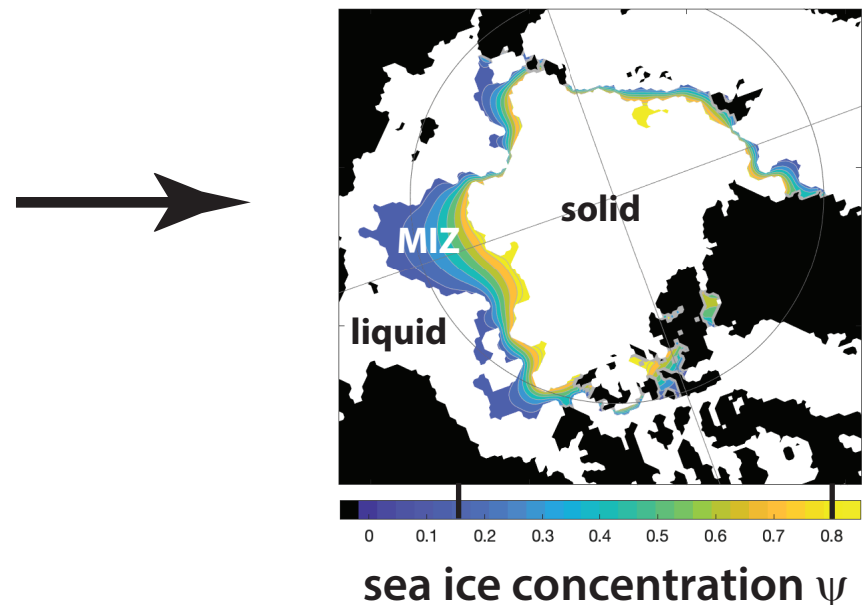
OBJECTIVE: model & predict dramatic annual cycle
impacts climate dynamics, polar ecology, human activities

mushy layer physics in the lab



NaCl-H₂O in lab
(Peppin et al., 2007, J. Fluid Mech.)

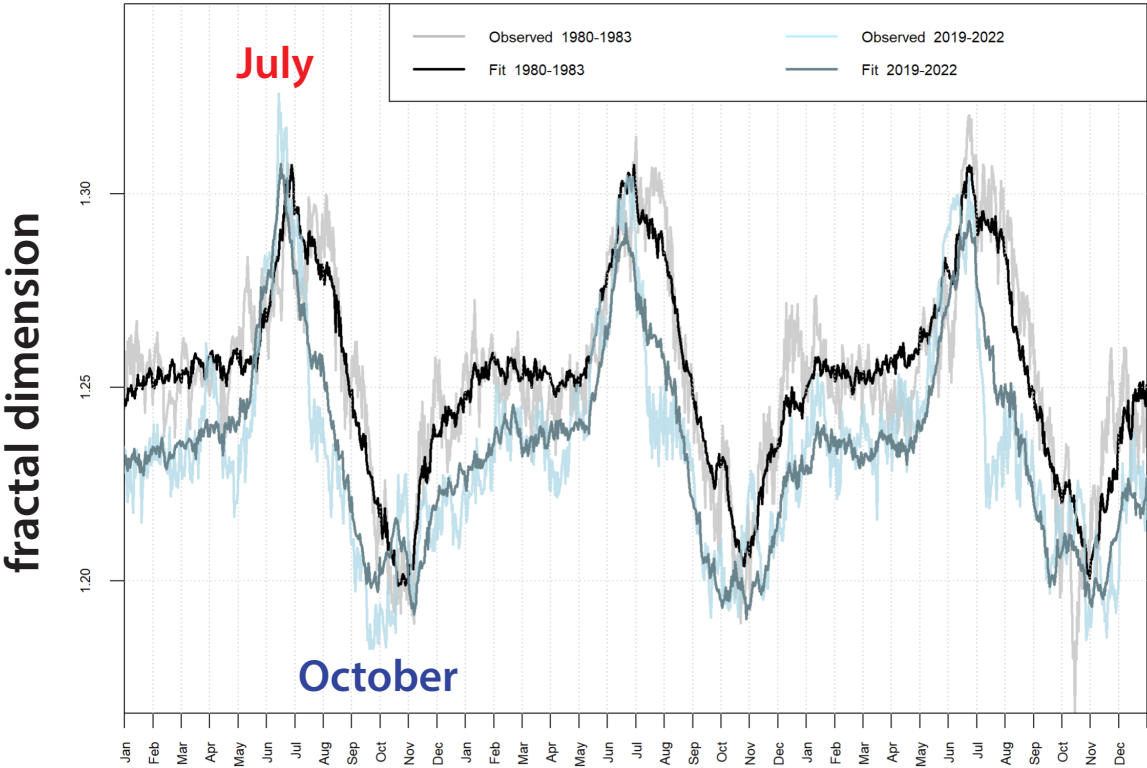
Arctic MIZ as a mushy layer



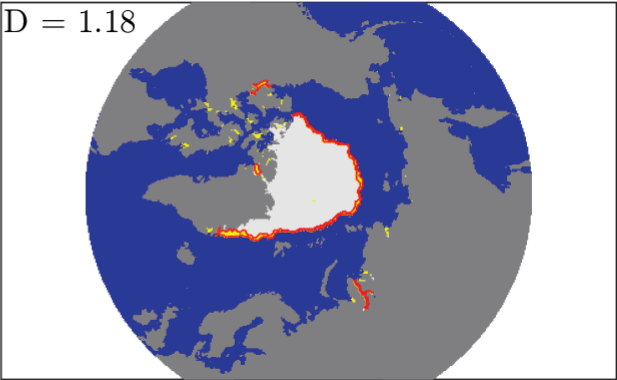
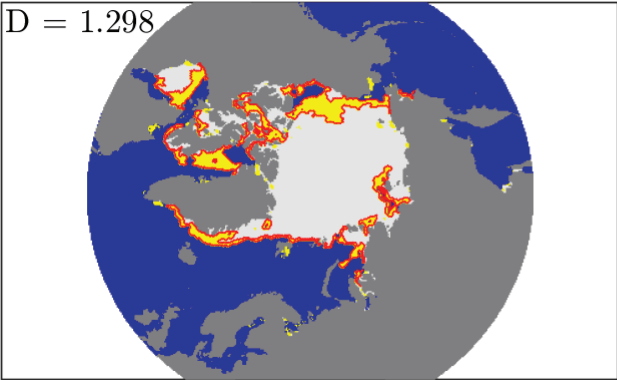
Identifying Fractal Geometry in Arctic Marginal Ice Zone Dynamics

Julie Sherman, Court Strong, Ken Golden, *Environ. Res. Lett.* 2025

Compute the fractal dimension of the boundary of the Arctic MIZ by boxcounting methods; analyze seasonal cycle and long term trends.



early summer



early autumn

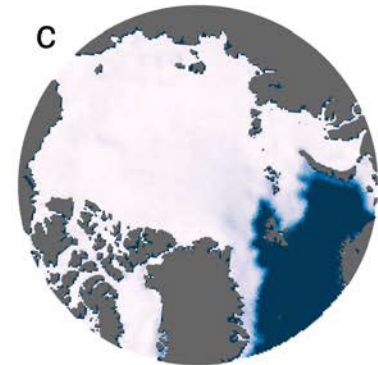
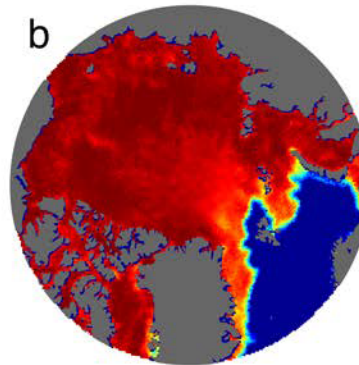
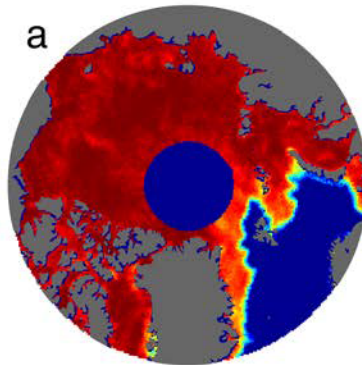
wave and thermal interactions with fractal boundary

Filling the polar data gap with partial differential equations

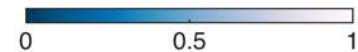
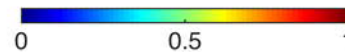
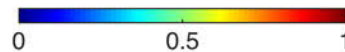
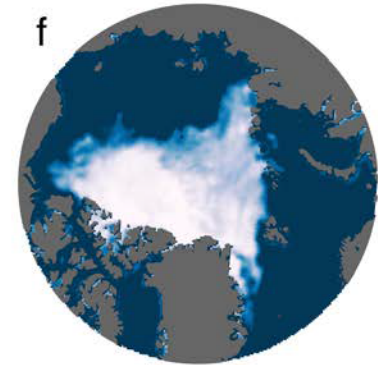
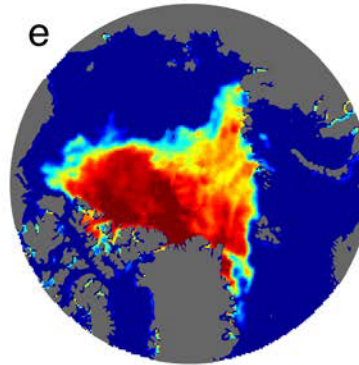
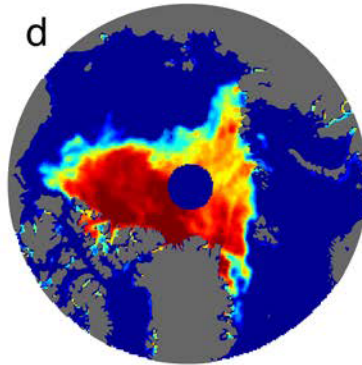
hole in satellite coverage
of sea ice concentration field

previously assumed
ice covered

Gap radius: 611 km
06 January 1985



Gap radius: 311 km
30 August 2007



$$\Delta\psi=0$$

fill = harmonic function satisfying
satellite BC's plus learned stochastic term

Strong and Golden, *Remote Sensing* 2016
Strong and Golden, *SIAM News* 2017

Global Sea Ice Concentration Climate Data Records, 2022
Lavergne, Sorensen, et al., Norwegian Met. Inst., ... OSI SAF



ISSN 0002-9920 (print)
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Notices

of the American Mathematical Society

November 2020

Volume 67, Number 10



Conclusions

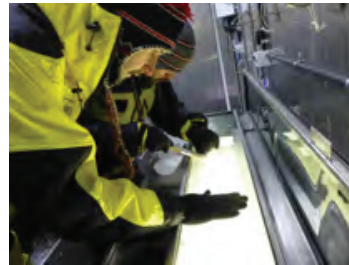
Our research is helping to improve projections of Earth's climate, its sea ice packs, and the fate of the ecosystems they support.

Cross pollination helps us study sea ice, while our sea ice math advances the theory of composites, inverse problems, etc.

Sea ice is really cool --- modeling it leads to unexpected areas of math and physics.

Mathematics is the operating system of science and engineering!

Thank you to so many postdocs, graduate students, undergraduates, high school students and colleagues who contributed to this work!



U. of Utah students in the Arctic and Antarctic (2003-2022): closing the gap between theory and observation - making math models come alive and experiencing climate change firsthand.

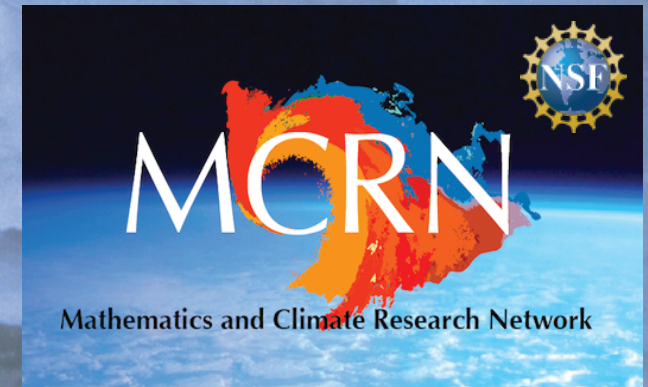
THANK YOU

Office of Naval Research

Applied and Computational Analysis Program
Arctic and Global Prediction Program

National Science Foundation

Division of Mathematical Sciences
Division of Polar Programs



Buchanan Bay, Antarctica Mertz Glacier Polynya Experiment July 1999

ICEBREAKER BURNS

Sydney Morning Herald
23 July, 1998

Fire strands Antarctic ship in sea ice

THE ADVERTISER (Adelaide)
Thurs 23 July 1998



