Mathematics 2210 Fall 2019

PRACTICE EXAM III SOLUTIONS

1. Evaluate the integral
$$\iint_R (x+y)dA,$$
where R is the triangle with vertices $(0,0),(0,4)$ and $(1,4)$.

Solution. The only difficulty is in determining the limits of integration. The triangle can be described by x going from 0 to 4 and y going from 0 to $1/4x$. So,
$$\iint_R (x+y)dA = \int_0^4\int_0^{x/4} (x+y)dydx$$
$$= \int_0^4 (xy + y^2/2)_0^{x/4} dx = \int_0^4 x^2/4 + x^2/32 dx$$
$$= (1/3)(9x^3/32)_0^4 = 6$$

2. Evaluate the iterated integral,
$$\int_0^2\int_0^{\sqrt{4-x^2}} (x+y)dydx.$$

Solution.
$$\int_0^2\int_0^{\sqrt{4-x^2}} (x+y)dydx = \int_0^2 (xy+y^2)_0^{\sqrt{4-x^2}} dx$$
$$= \int_0^2 x\sqrt{4-x^2} + 2 - x^2/2 dx = (-(1/2)(2/3)(4-x^2)^{3/2} + 2x - x^3/6)_0^2$$
$$= 16/3$$

3. Evaluate the following integral by changing to polar coordinates,
$$\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} xdxdy.$$

Solution. By drawing the line $x=y$ and circle $x=\sqrt{4-y^2}$ you can see that the region we are integrating over is the wedge of the circle of radius 2 in between angle 0 and $\pi/4$. Thus, changing the limits and integrand appropriately,

$$\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} xdxdy = \int_0^{\pi/4}\int_0^2 r\cos\theta r dr d\theta$$
$$= \int_0^{\pi/4}(r^3/3\cos\theta)_0^2 d\theta = \int_0^{\pi/4}(8/3\cos\theta)d\theta$$
$$= (8/3\sin\theta)_0^{\pi/4} = 4\sqrt{2}/3$$

4. Compute the surface area of the bottom part of the paraboloid $z = x^2 + y^2$ that is cut off by the plane $z = 9$.

   Solution. The region over which we are integrating is the circle $x^2 + y^2 = 9$ of radius 3, denoted by R. We begin with the formula for the surface area integral:

$$\int\int_R \sqrt{(2x)^2 + (2y)^2 + 1} dA$$

   The numbers and region suggest using polar coordinates:

$$\int_0^{2\pi}\int_0^3 r\sqrt{4r^2+1} dr d\theta$$
$$= \int_0^{2\pi}(2/3)(1/8)(4r^2+1)^{3/2}|_0^3 d\theta = \int_0^{2\pi}(1/12)(37^{3/2}-1)d\theta$$
$$= \pi/6(37^{3/2}-1)$$

5. Compute the surface area of the part of the sphere $x^2+y^2+z^2 = a^2$ inside the circular cylinder $x^2 + y^2 = b^2$, where $0 < b \le a$.

   Solution. The region R that we integrate over is a circle of radius b. Solving the surface equation for z, we get $z = \sqrt{a^2 - x^2 - y^2}$. The partials we will need for the surface area formula are

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{a^2 - x^2 - y^2}}$$
$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$$

   Now the surface area is

$$SA = \int\int_R \sqrt{\frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2} + 1} dA$$
$$= \int\int_R \sqrt{\frac{a^2}{a^2 - x^2 - y^2}} dA$$

   Converting now to polar coordinates,

$$= \int_0^{2\pi}\int_0^b \sqrt{\frac{a^2}{a^2-r^2}} r dr d\theta$$

$$= \int_0^{2\pi}\int_0^b a(a^2-r^2)^{-1/2} r dr d\theta$$

$$= \int_0^{2\pi} -a(a^2-r^2)^{1/2}|_0^b d\theta$$

$$= \int_0^{2\pi} -a(a^2-b^2)^{1/2} + a(a) d\theta = 2\pi(a^2 - a\sqrt{a^2-b^2})$$

Since this only gives us one of the two caps, multiply the answer by 2 to achieve the final answer.

$SA = 4\pi(a^2 - a\sqrt{a^2-b^2})$

6. Compute the volume of the solid in the first octant bounded by $y = 2x^2$ and $y + 4z = 8$.

Solution. The region in the xy plane that we are integrating over is defined by $y = 2x^2$, the y-axis, and the line $y = 8$ (since z must be positive in the second equation). Thus, solving the second equation for z, we integrate this "height" function over our region. The x-values run from 0 to 2.

$$= \int_0^2\int_{2x^2}^8 (2 - y/4) dy dx$$

$$= \int_0^2 (2y - y^2/8)_{2x^2}^8 dx$$

$$= \int_0^2 (16 - 8^2/8) - (4x^2 - 4x^4/8) dx$$

$$= (16x - 8x - 4x^3/3 - x^5/10)_0^2 = 16 - 32/3 + 16/5 = 128/3$$

7. Compute the Jacobian $J(r,\theta)$ of the transformation from polar coordinates to Cartesian coordinates given below:

$$x = r\cos\theta$$
$$y = r\sin\theta.$$

Solution.

$J(r,\theta) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r(\cos^2\theta + \sin^2\theta) = r$

8. Compute the Jacobian $J(x, y)$ of the transformation from Cartesian coordinates to polar coordinates given below:

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Recall: $D_x \tan^{-1} x = \frac{1}{1+x^2}$. What is the relationship between $J(r, \theta)$ and $J(x, y)$?

Solution.

$$J(x, y) = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x(1/2)(x^2 + y^2)^{-1/2} & 2y(1/2)(x^2 + y^2)^{-1/2} \\ \frac{1}{1+(y/x)^2}(-\frac{y}{x^2}) & \frac{1}{1+(y/x)^2}(1/x) \end{pmatrix}$$

$$= (x^2 + y^2)^{-1/2}(\tfrac{1}{1+(y/x)^2})(1 + (y/x)^2) = (x^2 + y^2)^{-1/2}.$$

This is the inverse transformation of the previous problem, so the Jacobian should be the inverse. Indeed, $(x^2 + y^2)^{-1/2} = 1/r$.

9. Let $u(x, y) = \log \sqrt{x^2 + y^2} = \log r$.

(a) Find the vector field associated with this scalar field, by computing $\operatorname{grad} u = \nabla u$.

Solution.

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2 + y^2}}(x)(x^2 + y^2)^{-1/2} = \frac{x}{x^2 + y^2}$$
$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

So $\nabla u = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$

(b) Compute $\operatorname{curl}(\operatorname{grad} u) = \nabla \times (\nabla u)$.

Solution. Using the formula for curl and a constant k-value of 0,

$$\operatorname{curl}(\operatorname{grad} u) = (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mathbf{k}$$
$$\operatorname{curl}(\operatorname{grad} u) = (\frac{-2xy}{(x^2 + y^2)^2} - \frac{-2xy}{(x^2 + y^2)^2})\mathbf{k} = 0$$

Since this was a conservative vector field (the gradient of a scalar field), this computation was actually unnecessary. The curl of a conservative vector field is always 0 (Theorem D, 14.3). On this test, you should still show the above computation since we have not covered this yet.

(c) What are the level sets?

Solution. Note that the direction of each vector is the same as the direction of (x, y). Thus at a point (x, y), the vector is pointing directly away from the origin.

The vectors are thus perpendicular to circles centered at the origin. These circles are the level curves.

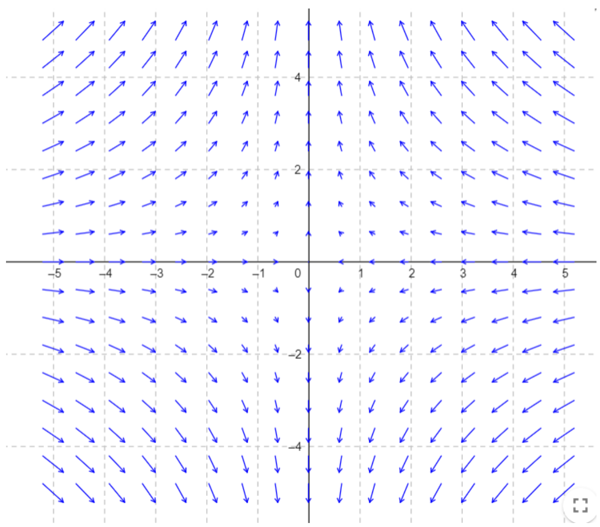
10. Let $\varphi(x, y) = x^2 - y^2$.

    (a) Compute $\vec{F} = -\operatorname{grad}\varphi = -\nabla\varphi$.

    Solution. $-\nabla\varphi = (-2x, 2y)$.

    (b) Sketch a diagram in the plane of the vector field $\vec{F}$.

    Solution.



    (c) Compute $\operatorname{div}(\operatorname{grad}\varphi) = \nabla \cdot (\nabla\varphi)$.

    Solution. $\nabla \cdot (\nabla\varphi) = 2 - 2 = 0$.

    (d) Based on your findings, what kind of function is φ?

    Solution. The divergence of the vector field is the same as the Laplacian of the function φ. As such, the divergence being 0 means φ is harmonic.

11. Find $\operatorname{div}\mathbf{F}$ and $\operatorname{curl}\mathbf{F}$, where $\mathbf{F}(x, y, z) = x^2\mathbf{i} - 2xy\mathbf{j} + yz^2\mathbf{k}$.

    Solution.

$$\operatorname{div}\mathbf{F} = 2x - 2x + 2yz = 2yz.$$
$$\operatorname{curl}\mathbf{F} = (z^2 - 0)\mathbf{i} + (0 - 0)\mathbf{j} + (-2y - 0)\mathbf{k} = z^2\mathbf{i} - 2y\mathbf{k}$$

12. Find the volume of a spherical ball of radius a using a triple integral.

Solution.
The equation for the outer edge of a sphere of radius a is given by $x^2 + y^2 + z^2 = a^2$. If we want to consider the volume inside, then we are considering the regions $x^2 + y^2 + z^2 \leq a^2$. We will set up the inequalities in three ways.

1. *In Cartesian Coordinates*: Solving for z gives $-\sqrt{a^2 - x^2 - y^2} \leq z \leq \sqrt{a^2 - x^2 - y^2}$. Then the projection of the sphere onto the xy-plane (i.e. the equation you get when you have $z = 0$ in the sphere equation) is just the circle $x^2 + y^2 = a^2$. Now we must describe this with inequalities. All together, the solid can be described by the inequalities $-a \leq x \leq a$, $-\sqrt{a^2 - x^2} \leq y \leq \sqrt{a^2 - x^2}$, $-\sqrt{a^2 - x^2 - y^2} \leq z \leq \sqrt{a^2 - x^2 - y^2}$. So we can find the volume:

$$\iiint_E 1\, dV = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} 1\, dzdydx = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2 - x^2 - y^2}\, dydx$$

$$= \int_{-a}^{a} 2\frac{1}{2}\pi(a^2 - x^2)\, dx = \pi(2a^3 - \frac{2}{3}a^3) = \frac{4}{3}\pi a^3.$$

Note: Same note as I made for the circular cylinder concerning skipped steps in the integration.

2. *In Cylindrical Coordinates*: The bound on z would still be the same, but we would use polar for x and y. All together, the solid can be described by $0 \leq \theta \leq 2\pi$, $0 \leq r \leq a$, $-\sqrt{a^2 - r^2} \leq z \leq \sqrt{a^2 - r^2}$. And we get a volume of:

$$\iiint_E 1\, dV = \int_0^{2\pi} \int_0^a \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\, dzdrd\theta = 2\pi \int_0^a 2r\sqrt{a^2 - r^2}\, dr$$

$$= 2\pi \int_0^{a^2} \sqrt{u}\, du = 2\pi\frac{2}{3}a^3 = \frac{4}{3}\pi a^3$$

3. *In Spherical Coordinates*: In spherical coordinates, the sphere is all points where $0 \leq \phi \leq \pi$ (the angle measured down from the positive z axis ranges), $0 \leq \theta \leq 2\pi$ (just like in polar coordinates), and $0 \leq \rho \leq a$. And we get a volume of:

$$\iiint_E 1\, dV = \int_0^{\pi} \int_0^{2\pi} \int_0^a \rho^2 \sin(\phi) d\rho d\theta d\phi = \int_0^{\pi} \sin(\phi)\, d\phi \int_0^{2\pi} d\theta \int_0^a \rho^2 d\rho = (2)(2\pi)\left(\frac{1}{3}a^3\right) = \frac{4}{3}\pi a^3$$

In all three cases, we see that we get the expected volume formula.

13. Find the mass of a cylinder of radius a and height h if its mass density is proportional to the distance to its base.

Solution. We know the area of each circle slice is πa^2. We have that the mass is proportional to the height. So, each slice mass is going to be $kz\pi a^2 dz$ for some proportionality constant k and we integrate from 0 to h. Finally, for the mass M of the cylinder we have,

$$M = \int_0^h kz\pi a^2 dz = \frac{k\pi a^2 h^2}{2}.$$

Or,

$$M = \int_0^h \int_0^{2\pi} \int_0^a kz\, r\, dr\, d\theta\, dz = \frac{k\pi a^2 h^2}{2}.$$