

**Mathematics 2210**                      **Fall 2013**  
**PRACTICE EXAM III SOLUTIONS**

1. Evaluate the integral

$$\iint_R (x + y) dA,$$

where  $R$  is the triangle with vertices  $(0, 0)$ ,  $(0, 4)$  and  $(1, 4)$ .

**Solution.** The only difficulty is in determining the limits of integration. The triangle can be described by  $x$  going from 0 to 4 and  $y$  going from 0 to  $1/4x$ . So,

$$\begin{aligned} \iint_R (x + y) dA &= \int_0^4 \int_0^{x/4} (x + y) dy dx \\ &= \int_0^4 (xy + y^2/2)_0^{x/4} dx = \int_0^4 x^2/4 + x^2/32 dx \\ &= (1/3)(9x^3/32)_0^4 = 6 \end{aligned}$$

2. Evaluate the iterated integral,

$$\int_0^2 \int_0^{\sqrt{4-x^2}} (x + y) dy dx.$$

**Solution.**

$$\begin{aligned} \int_0^2 \int_0^{\sqrt{4-x^2}} (x + y) dy dx &= \int_0^2 (xy + y^2)_0^{\sqrt{4-x^2}} dx \\ &= \int_0^2 x\sqrt{4-x^2} + 2 - x^2/2 dx = -(1/2)(2/3)(4-x^2)^{3/2} + 2x - x^3/6)_0^2 \\ &= 16/3 \end{aligned}$$

3. Evaluate the following integral by changing to polar coordinates,

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} x dx dy.$$

**Solution.** By drawing the line  $x = y$  and circle  $x = \sqrt{4-y^2}$  you can see that the region we are integrating over is the wedge of the circle of radius 2 in between angle 0 and  $\pi/4$ . Thus, changing the limits and integrand appropriately,

$$\begin{aligned}
\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} x dx dy &= \int_0^{\pi/4} \int_0^2 r \cos \theta r dr d\theta \\
&= \int_0^{\pi/4} (r^3/3 \cos \theta)_0^2 d\theta = \int_0^{\pi/4} (8/3 \cos \theta) d\theta \\
&= (8/3 \sin \theta)_0^{\pi/4} = 4\sqrt{2}/3
\end{aligned}$$

4. Compute the surface area of the bottom part of the paraboloid  $z = x^2 + y^2$  that is cut off by the plane  $z = 9$ .

**Solution.** The region over which we are integrating is the circle  $x^2 + y^2 = 9$  of radius 3, denoted by  $R$ . We begin with the formula for the surface area integral:

$$\int \int_R \sqrt{(2x)^2 + (2y)^2 + 1} dA$$

The numbers and region suggest using polar coordinates:

$$\begin{aligned}
&\int_0^{2\pi} \int_0^3 r \sqrt{4r^2 + 1} dr d\theta \\
&= \int_0^{2\pi} (2/3)(1/8)(4r^2 + 1)^{3/2} \Big|_0^3 d\theta = \int_0^{2\pi} (1/12)(37^{3/2} - 1) d\theta \\
&= \pi/6(37^{3/2} - 1)
\end{aligned}$$

5. Compute the surface area of the part of the sphere  $x^2 + y^2 + z^2 = a^2$  inside the circular cylinder  $x^2 + y^2 = b^2$ , where  $0 < b \leq a$ .

**Solution.** The region  $R$  that we integrate over is a circle of radius  $b$ . Solving the surface equation for  $z$ , we get  $z = \sqrt{a^2 - x^2 - y^2}$ . The partials we will need for the surface area formula are

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{-x}{\sqrt{a^2 - x^2 - y^2}} \\
\frac{\partial z}{\partial y} &= \frac{-y}{\sqrt{a^2 - x^2 - y^2}}
\end{aligned}$$

Now the surface area is

$$\begin{aligned}
SA &= \int \int_R \sqrt{\frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2} + 1} dA \\
&= \int \int_R \sqrt{\frac{a^2}{a^2 - x^2 - y^2}} dA
\end{aligned}$$

Converting now to polar coordinates,

$$\begin{aligned}
&= \int_0^{2\pi} \int_0^b \sqrt{\frac{a^2}{a^2 - r^2}} r dr d\theta \\
&= \int_0^{2\pi} \int_0^b a(a^2 - r^2)^{-1/2} r dr d\theta \\
&= \int_0^{2\pi} -a(a^2 - r^2)^{1/2} \Big|_0^b d\theta \\
&= \int_0^{2\pi} -a(a^2 - b^2)^{1/2} + a(a) d\theta = 2\pi(a^2 - a\sqrt{a^2 - b^2})
\end{aligned}$$

Since this only gives us one of the two caps, multiply the answer by 2 to achieve the final answer.

$$SA = 4\pi(a^2 - a\sqrt{a^2 - b^2})$$

6. Compute the volume of the solid in the first octant bounded by  $y = 2x^2$  and  $y+4z = 8$ .

**Solution.** The region in the  $xy$  plane that we are integrating over is defined by  $y = 2x^2$ , the  $y$ -axis, and the line  $y = 8$  (since  $z$  must be positive in the second equation). Thus, solving the second equation for  $z$ , we integrate this "height" function over our region. The  $x$ -values run from 0 to 2.

$$\begin{aligned}
&= \int_0^2 \int_{2x^2}^8 (2 - y/4) dy dx \\
&= \int_0^2 (2y - y^2/8) \Big|_{2x^2}^8 dx \\
&= \int_0^2 (16 - 8^2/8) - (4x^2 - 4x^4/8) dx \\
&= (16x - 8x - 4x^3/3 - x^5/10) \Big|_0^2 = 16 - 32/3 + 16/5 = 128/3
\end{aligned}$$

7. Compute the Jacobian  $J(r, \theta)$  of the transformation from polar coordinates to Cartesian coordinates given below:

$$\begin{aligned}
x &= r \cos \theta \\
y &= r \sin \theta.
\end{aligned}$$

**Solution.**

$$J(r, \theta) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} = r(\cos^2 \theta + \sin^2 \theta) = r$$

8. Compute the Jacobian  $J(x, y)$  of the transformation from Cartesian coordinates to polar coordinates given below:

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1} \left( \frac{y}{x} \right)$$

Recall:  $D_x \tan^{-1} x = \frac{1}{1+x^2}$ . What is the relationship between  $J(r, \theta)$  and  $J(x, y)$ ?

**Solution.**

$$J(x, y) = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x(1/2)(x^2 + y^2)^{-1/2} & 2y(1/2)(x^2 + y^2)^{-1/2} \\ \frac{1}{1+(y/x)^2} \left( -\frac{y}{x^2} \right) & \frac{1}{1+(y/x)^2} (1/x) \end{pmatrix}$$

$$= (x^2 + y^2)^{-1/2} \left( \frac{1}{1+(y/x)^2} \right) (1 + (y/x)^2) = (x^2 + y^2)^{-1/2}.$$

This is the inverse transformation of the previous problem, so the Jacobian should be the inverse. Indeed,  $(x^2 + y^2)^{-1/2} = 1/r$ .

9. Let  $u(x, y) = \log \sqrt{x^2 + y^2} = \log r$ .

- (a) Find the vector field associated with this scalar field, by computing  $\text{grad } u = \nabla u$ .

**Solution.**

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2 + y^2}}(x)(x^2 + y^2)^{-1/2} = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

$$\text{So } \nabla u = \left( \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right)$$

- (b) Compute  $\text{curl}(\text{grad } u) = \nabla \times (\nabla u)$ .

**Solution.** Using the formula for curl and a constant k-value of 0,

$$\text{curl}(\text{grad } u) = \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}$$

$$\text{curl}(\text{grad } u) = \left( \frac{-2xy}{(x^2 + y^2)^2} - \frac{-2xy}{(x^2 + y^2)^2} \right) \mathbf{k} = 0$$

Since this was a conservative vector field (the gradient of a scalar field), this computation was actually unnecessary. The curl of a conservative vector field is always 0 (Theorem D, 14.3). On this test, you should still show the above computation since we have not covered this yet.

- (c) What are the level sets?

**Solution.** Note that the direction of each vector is the same as the direction of  $(x, y)$ . Thus at a point  $(x, y)$ , the vector is pointing directly away from the origin.

The vectors are thus perpendicular to circles centered at the origin. These circles are the level curves.

10. Let  $\varphi(x, y) = x^2 - y^2$ .

(a) Compute  $\text{grad } \varphi = \nabla \varphi$ .

**Solution.**  $\nabla \varphi = (2x, -2y)$ .

(b) Compute  $\text{div}(\text{grad } \varphi) = \nabla \cdot (\nabla \varphi)$ .

**Solution.**  $\nabla \cdot (\nabla \varphi) = 2 - 2 = 0$ .

(c) Based on your findings, what kind of function is  $\varphi$ ?

**Solution.** The divergence of the vector field is the same as the Laplacian of the function  $\varphi$ . As such, the divergence being 0 means  $\varphi$  is harmonic.

11. Find  $\text{div } \mathbf{F}$  and  $\text{curl } \mathbf{F}$ , where  $\mathbf{F}(x, y, z) = x^2\mathbf{i} - 2xy\mathbf{j} + yz^2\mathbf{k}$ .

**Solution.**

$$\begin{aligned}\text{div } \mathbf{F} &= 2x - 2x + 2yz = 2yz. \\ \text{curl } \mathbf{F} &= (z^2 - 0)\mathbf{i} + (0 - 0)\mathbf{j} + (-2y - 0)\mathbf{k} = z^2\mathbf{i} - 2y\mathbf{k}\end{aligned}$$

12. Evaluate the following line integral, where  $C$  is the curve  $x = 3t, y = t^3, 0 \leq t \leq 1$ .

$$\int_C (x^3 + y) ds$$

**Solution.**  $dx/dt = 3, dy/dt = 3t^2$ .

$$\begin{aligned}\int_C (x^3 + y) ds &= \int_0^1 ((3t)^3 + t^3) \sqrt{3^2 + 9t^4} dt \\ &= \int_0^1 28t^3(3) \sqrt{1 + t^4} dt = 28(3)(2/3)(1/4)(1 + t^4)^{3/2} \Big|_0^1 = 14(2^{3/2} - 1)\end{aligned}$$

13. Evaluate the following line integral, where  $C$  is the line segment from  $(0, 0)$  to  $(\pi, 2\pi)$ .

$$\int_C (\sin x + \cos y) ds$$

**Solution.** Parameterize  $C$  by  $x = t, y = 2t, 0 \leq t \leq \pi$ .  $dx/dt = 1, dy/dt = 2$ .

$$\begin{aligned} \int_C (\sin x + \cos y) ds &= \int_0^\pi (\sin t + \cos 2t) \sqrt{1 + 4} dt \\ &= \sqrt{5} (-\cos t + 1/2 \sin 2t) \Big|_0^\pi = \sqrt{5} (1 - (-1)) = 2\sqrt{5} \end{aligned}$$