Mathematics 2210 Practice Exam II SOLUTIONS Fall 2019

1. Suppose you have a function $z = f(x, y)$ whose graph is a surface in $\mathbb{R}^3$. Describe how the level sets of the function relate geometrically to the surface. What is the relationship between the level sets and the gradient of f, ∇f?

   Solution. The level sets of a function $f(x, y)$ will be curves in a plane. They can be thought of as points (x, y) in the plane that have the same z-value. They also form contour lines. The directional derivative in the direction of the line is always 0.

   The level sets are related to the gradient in that a vector perpendicular to the level set curve at a point is in the direction of the gradient at that point. One can think of contour lines on a map, and that the steepest direction at a point is the one perpendicular to the contour line at that point.

2. Consider the paraboloid defined by $z = f(x, y) = (x - 2)^2 + (y - 2)^2$.

   (a) Sketch the paraboloid.

   (b) On a separate set of xy axes, sketch the level curves $z = 1$ and $z = \sqrt{2}$.

   (c) On the same axes as above, draw the gradient vector at the point $(2, 0)$.

   (d) Find the global extrema of f on $\mathbb{R}^2$ and verify your results using the second partial derivative test.

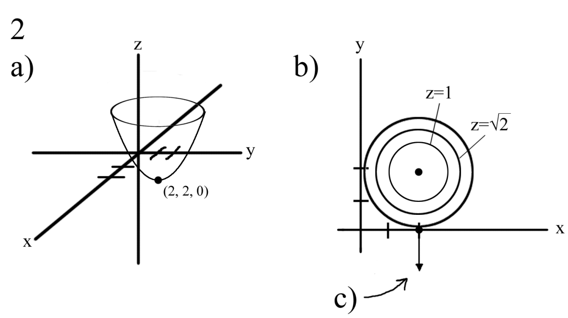

Figure 1: The circles depicted in the diagram for (b) have radii 1, $\sqrt[4]{2}$, and 2 respectively.

   (d) Take the gradient of f and set it equal to 0 to find possible locations of extrema. $\nabla f = (2(x - 2), 2(y - 2))$ is 0 when $(x, y) = (2, 2)$. From the picture in (a), it seems f has a global minimum at $(2, 2)$. We need to check this result with the second partial derivative test. $f_{xx} = 2, f_{yy} = 2, f_{xy} = 0 = f_{yx}$. So the Hessian matrix looks like $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. The determinant of the Hessian is $4 > 0$, and $f_{xx} = 2 > 0$. Thus, by the second partial derivative test, $(2, 2)$ is a minimum point.

3. Suppose that the temperature in $\mathbb{R}^3$ is given by

$$T(x, y, z) = \frac{1}{1 + x^2 + y^2 + z^2},$$

and further suppose that your position is given by the curve:

$$\mathbf{r}(t) = (x(t), y(t), z(t)) = (2t, 4t^2, 1).$$

(a) Use the chain rule to find the rate of change $\frac{dT}{dt}$ of the temperature T with respect to time t, as you travel along the curve given above. Express your answer in terms of t only and simplify it.

Solution.
$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}$
$\frac{dT}{dt} = (-1)(2x)(1+x^2+y^2+z^2)^{-2}(2) + (-1)(2y)(1+x^2+y^2+z^2)^{-2}(8t) + (-1)(2z)(1+x^2+y^2+z^2)^{-2}(0)$
$\frac{dT}{dt} = -8t(2+4t^2+16t^4)^{-2} - 64t^3(2+4t^2+16t^4)^{-2}$

(b) Find the direction in which the temperature is increasing the fastest at time $t = 2$.

Solution. The position at $t = 2$ is $r(2) = (4, 16, 1)$. The gradient of T is
$(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z})$
$= ((-1)(2x)(1+x^2+y^2+z^2)^{-2}, (-1)(2y)(1+x^2+y^2+z^2)^{-2}, (-1)(2z)(1+x^2+y^2+z^2)^{-2})$
$\nabla T(4, 16, 1) = (\frac{-8}{274^2}, \frac{-32}{274^2}, \frac{-2}{274^2})$
We are interested in the direction of this vector, which is the same as the direction of $(-8, -32, -2)$.
Direction $= 1/\sqrt{1092}(-8, -32, -2)$

4. Consider the function $f(x, y) = x^2 - xy^3$.

(a) If $x = \cos(t)$ and $y = \sin(t)$, find $\frac{df}{dt}$.

Solution. By the Chain Rule,
$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$
$\frac{df}{dt} = (2x - y^3)(-\sin(t)) + (-3xy^2)(\cos(t))$
$\frac{df}{dt} = (2(\cos(t)) - (\sin(t))^3)(-\sin(t)) + (-3(\cos(t))(\sin(t))^2)(\cos(t))$

(b) Find the differential df at the point $(1, 1)$ if x increases by 0.1 and y decreases by 0.2.

Solution. The formula we need is
$df = (\frac{\partial f}{\partial x})dx + (\frac{\partial f}{\partial y})dy$
$df = (2x - y^3)dx + (-3xy^2)dy$
$df(1, 1) = (2 - 1)0.1 + (-3)(-0.2)$
$df(1, 1) = 0.7$

5. Find the following limit. If it does not exist, demonstrate why not.

$$\lim_{(x,y)\to(0,0)} \frac{x - 7y}{x + y}$$

Solution. Approaching (0, 0) along the line $y = 0$, the limit is

$$\lim_{x \to 0} \frac{x}{x} = 1$$

Approaching (0, 0) along the line $x = 0$, the limit is

$$\lim_{y \to 0} \frac{-7y}{y} = -7$$

The two limits do not agree so the original limit does not exist.

6. Find the following limit. If it does not exist, demonstrate why not.

$$\lim_{(x,y) \to (0,0)} \frac{x^2}{y}$$

Solution. Along the line $x = y$, the limit is

$$\lim_{x \to 0} \frac{x^2}{x} = 0$$

Along the curve $x = \sqrt{y}$, the limit is

$$\lim_{y \to 0} \frac{y}{y} = 1$$

The two limits do not agree so the original limit does not exist.

7. Find the following limit. If it does not exist, demonstrate why not.

$$\lim_{(x,y) \to (0,0)} \frac{1 - \cos\sqrt{x^2 + y^2}}{x^2 + y^2}$$

Solution. Convert to polar coordinates to simplify the computation. Set $x = r\cos\theta, y = r\sin\theta$. Using this and L'Hopital's Rule, we arrive at

$$\lim_{(x,y) \to (0,0)} \frac{1 - \cos\sqrt{x^2 + y^2}}{x^2 + y^2} = \lim_{r \to 0} \frac{1 - \cos r}{r^2} = \lim_{r \to 0} \frac{\sin r}{2r} = \lim_{r \to 0} \frac{\cos r}{2} = \frac{1}{2}.$$

8. Find the directional derivative of $f(x, y, z) = (x^2 - y^2)e^{2z}$,

(a) at the point $P = (1, 2, 0)$ in the direction $2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

Solution. The unit direction is $\vec{u} = \frac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} + \mathbf{k})$.
The gradient of f is $\nabla f = 2xe^{2z}\mathbf{i} - 2ye^{2z}\mathbf{j} + 2(x^2 - y^2)e^{2z}\mathbf{k}$.
The directional derivative is
$D_{\vec{u}}f = \nabla f(1, 2, 0) \cdot \vec{u}$
$D_{\vec{u}}f = (2\mathbf{i} - 4\mathbf{j} - 6\mathbf{k}) \cdot \frac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} + \mathbf{k})$
$D_{\vec{u}}f = \frac{1}{\sqrt{6}}(4 - 4 - 6) = -\frac{6}{\sqrt{6}}$.

(b) At the point P, find the direction of maximal increase of f.

Solution. The direction of maximal increase is the direction of the gradient.
$||\nabla f(1, 2, 0)|| = \sqrt{4 + 16 + 36} = 2\sqrt{14}$.
Unit Vector $= \frac{1}{2\sqrt{14}}(2\mathbf{i} - 4\mathbf{j} - 6\mathbf{k})$

9. Consider the surface defined by $f(x, y, z) = xe^y + ye^z + ze^x = 0$.

(a) Find the gradient of f.

Solution. $\nabla f(x, y, z) = (e^y + ze^x)\mathbf{i} + (xe^y + e^z)\mathbf{j} + (ye^z + e^x)\mathbf{k}$

(b) Find the equation for the tangent plane at the point $(0, 0, 0)$.

Solution. The equation for the tangent plane has coefficients equal to the components of the gradient at (0, 0, 0).
$\nabla f(0, 0, 0) = (1)\mathbf{i} + (1)\mathbf{j} + (1)\mathbf{k}$
Thus the tangent plane has the form $x + y + z = C$. Since (0, 0, 0) is on the plane, $C = 0$. Thus the tangent plane is $x + y + z = 0$.

(c) Find the directional derivative of f in the direction $\mathbf{i} + \mathbf{j}$ at the point $(0, 0, 0)$.

Solution. The unit direction is $\vec{u} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$.
The directional derivative is
$D_{\vec{u}}f = \nabla f(0, 0, 0) \cdot \vec{u}$
$D_{\vec{u}}f = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$
$D_{\vec{u}}f = \frac{1}{\sqrt{2}}(1 + 1) = \sqrt{2}$.

10. Find the local maxima, minima, and saddle points of the function $f(x, y) = x^2 + y^2 - 3xy$.

Solution. First find the critical points, for which both f_x and f_y must be 0 at the same time.
$f_x = 2x - 3y$
$f_y = 2y - 3x$

Setting both equal to 0 and solving gives a system with a single solution, at (0, 0).

Now to use the Second Partial Derivative test we need to find the second partial derivates:

$f_{xx} = 2, f_{xy} = -3, f_{yy} = 2$

$D = f_{xx}((0,0))f_{yy}((0,0)) - (f_{xy}((0,0)))^2$

$D = (2)(2) - (-3)^2 = -5$

Since this is negative, the point is a saddle point.

11. Show that $u(x,t) = \cos(x - ct) + \sin(x - ct)$ solves the wave equation:

$$c^2\, u_{xx} = u_{tt} \qquad \text{OR} \qquad c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$$

Solution. We find u_{xx} and u_{tt} and make sure they solve the equation above.

$u_x = -\sin(x - ct) + \cos(x - ct)$

$u_{xx} = -\cos(x - ct) - \sin(x - ct)$

$u_t = c\sin(x - ct) - c\cos(x - ct)$

$u_{tt} = -c^2\cos(x - ct) - c^2\sin(x - ct)$

Plugging into the wave equation,

$c^2(-\cos(x - ct) - \sin(x - ct)) = -c^2\cos(x - ct) - c^2\sin(x - ct)$

The equation holds so u satisfies the wave equation.

12. Consider the saddle function $f(x,y) = x^2 - y^2$.

(a) Show that this function is harmonic.

Solution. A harmonic function must sastify Laplace's equation, $\nabla^2 f = 0$, or rewritten, $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$

$\frac{\partial^2 f}{\partial x^2} = 2$

$\frac{\partial^2 f}{\partial y^2} = -2$

$2 - 2 = 0.$

(b) Now consider this function on the unit disk $D = \{(x,y) : x^2 + y^2 \leq 1\}$. Find the global extrema of f on the disk D.

Solution. A harmonic function on a closed, bounded set always achieves its extrema on the boundary, in the case the unit circle $x^2 + y^2 = 1$. Changing f to polar coordinates, $f = (r\cos\theta)^2 - (r\sin\theta)^2$

$f = r^2(\cos^2\theta - \sin^2\theta)$
$f = r^2(2\cos^2\theta - 1)$

The unit circle in polar coordinates is the equation $r = 1$, thus we want to find the extrema of f over all values of θ.

$$f = 2\cos^2\theta - 1$$

This is maximized when $\cos\theta = 1$ or -1, so when $\theta = 0$ or π. The maximum attained at these points (the points $(-1, 0)$ and $(1, 0)$) is 1.
This is minimized when $\cos\theta = 0$, so when $\theta = \pi/2$ or $3\pi/2$. The minimum attained at these points (the points $(0, 1)$ and $(0, -1)$) is -1.

13. Let $\phi(x, y)$ be the electric potential due to a point charge in two dimensions, that is, $\phi(x, y) = k \ln r$, where $r = \sqrt{x^2 + y^2}$ and you may take $k = -1$. (a) Find the level curves of ϕ and its gradient $\vec{E} = -\nabla\phi$. Sketch $\vec{E}$ at the points $(1, 0), (0, 1), (-1, 0), (0, -1)$ and interpret its meaning. (b) Find the level sets for $\phi(x, y, z) = mgz$ in three dimensions, find $\vec{F} = -\nabla\phi$, and interpret the meaning of $\vec{F}$.

Solution. (a) To find the level curves of ϕ, we set $\phi(x, y) = k \ln r = c$, for some constant $c \in \mathbb{R}$; thus, $r = e^{c/k}$, so r is constant for the level curves of ϕ, that is, the level curves are circles. Further,

$$\begin{aligned}\vec{E} = -\nabla\phi &= -\Big(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\Big) \\ &= -\Big(k\frac{1}{\sqrt{x^2+y^2}}\frac{x}{\sqrt{x^2+y^2}}, k\frac{1}{\sqrt{x^2+y^2}}\frac{y}{\sqrt{x^2+y^2}}\Big) \\ &= -\frac{k}{x^2+y^2}(x, y).\end{aligned}$$

For $k = -1$, $\vec{E}$ is pointing outward, perpendicular to the unit circle, at $(1, 0), (0, 1), (-1, 0), (0, -1)$. Since the negative gradient of the electric potential is the electric field due to the point charge, the interpretation is that a second point charge in the plane would be driven outward by the point charge at the origin.

(b) We set $\phi(x, y, z) = mgz = c$, for a constant $c \in \mathbb{R}$; thus $z = \frac{c}{mg}$, that is, z is a constant. We conclude that the level sets are the planes where z is constant, that is those planes parallel to $z = 0$.

$$\begin{aligned}\vec{F} = -\nabla\phi &= -\Big(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\Big) \\ &= (0, 0, -mg).\end{aligned}$$

So $\vec{F}$ is a constant vector pointing downward, which is orthogonal to the level curves. Note that ϕ is the gravitational potential energy, so $\vec{F}$ is the gravitational field, the force on an object of mass m due to gravity.