Mathematics 2210−1,2 **Fall 2019**

PRACTICE EXAM I SOLUTIONS

1. Let $\vec{u} = (2, 2)$ and $\vec{v} = (3, -1)$. Find $\vec{u} + \vec{v}$ and illustrate this vector addition with a diagram in the plane, showing $\vec{u}$, $\vec{v}$ and the resultant vector. Illustrate multiplication by a scalar with a diagram showing $\vec{u}$, $3\vec{u}$, and $-\vec{u}$.

SOLUTION.

$\vec{u} + \vec{v} = (5, 1)$.

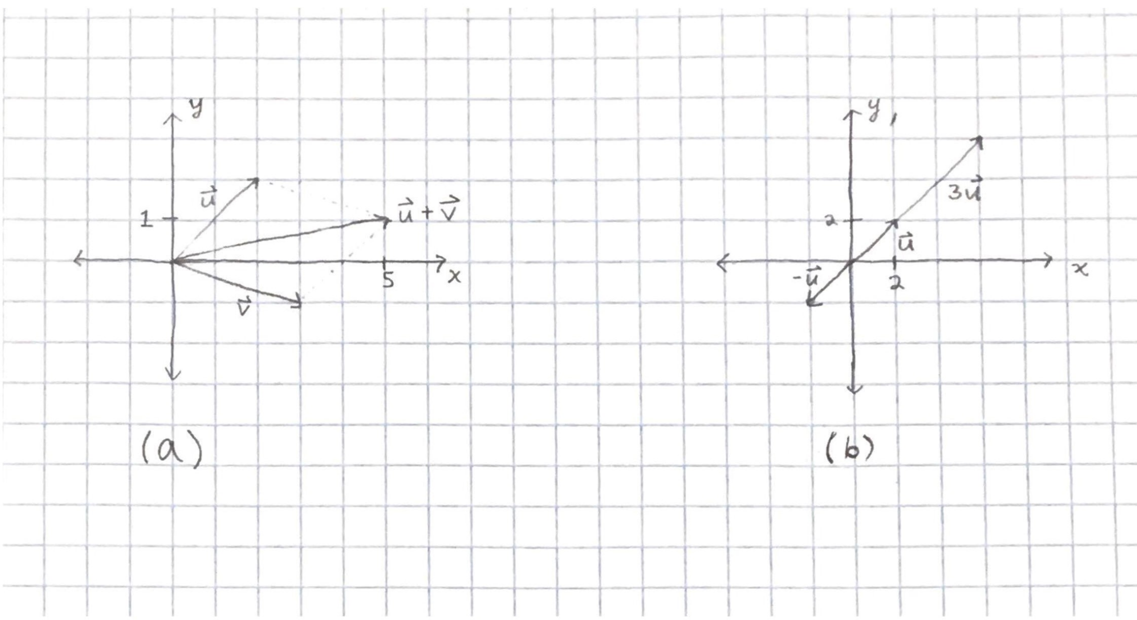

Figure 1: (a) illustrates $\vec{u} + \vec{v}$ and (b) illustrates $3\vec{u}$ and $-\vec{u}$.

2. Consider the vectors $\vec{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\vec{v} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$.

   (a) Find the length of $\vec{u}$.

   (b) Find $\vec{N} = \vec{u} \times \vec{v}$.

   (c) Find the cartesian equation of the plane with normal $\vec{N}$ through the point $P_0 = (1, 0, -1)$.

   (d) Find the vector projection of $\vec{v}$ onto $\vec{u}$.

SOLUTION.

a) $||\vec{u}|| = \sqrt{1 + 4 + 1} = \sqrt{6}$.

b) $\vec{u} \times \vec{v} = -2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} - (-6\mathbf{k} - 4\mathbf{i} + \mathbf{j}) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

c) $2x + 2y + 2z = C$
$2(1) + 0 + 2(-1) = C = 0$
$2x + 2y + 2z = 0$

d) $\frac{\vec{v} \cdot \vec{u}}{||u||^2} \vec{u} =$

$\frac{3+8+1}{1+4+1}(1, -2, 1) = (2, -4, 2)$

3. Determine the area of the triangle with vertices $\vec{P} = (0, 0), \vec{Q} = (3, 2), \vec{R} = (1, 4)$.

SOLUTION.

Use $\vec{PQ} = \vec{Q} - \vec{P} = (3, 2)$ as the base of the triangle, with length $\sqrt{13}$. Find the height of the triangle by finding the length of the orthogonal projection of $\vec{PR} = (1, 4)$ onto $\vec{PQ}$. First find the vector projection:

$$\frac{(1, 4) \cdot (3, 2)}{||(3, 2)||}(3, 2) = \frac{3 + 8}{4 + 9}(3, 2)$$
$$= (33/13, 22/13)$$

The orthogonal projection is $(1, 4)$ minus this vector:

$$(1, 4) - (33/13, 22/13) = (-20/13, 30/13)$$
$$||(-20/13, 30/13)|| = \sqrt{400/169 + 900/169} = \sqrt{1300/169} = 10/13\sqrt{13}$$

Thus the triangle's area is $\frac{1}{2}\sqrt{13}\frac{10}{13}\sqrt{13} = 5$.

4. Consider $\vec{e_1} = (\sqrt{3}/2, 1/2)$ and $\vec{e_2} = (-1/2, \sqrt{3}/2)$. Show that $\vec{e_1}$ and $\vec{e_2}$ each have unit length and that they are orthogonal. Rewrite the vector $\vec{v} = 4\mathbf{i} + 5\mathbf{j}$ in the orthonormal basis $\vec{e_1} = (\sqrt{3}/2, 1/2), \vec{e_2} = (-1/2, \sqrt{3}/2)$. In other words, expand or write $\vec{v}$ as $\vec{v} = a\vec{e_1} + b\vec{e_2}$ where a and b are scalar values.

SOLUTION.

First compute the magnitude of each vector.

$$||\vec{e_1}|| = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$$
$$||\vec{e_2}|| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

To show orthogonality, let's use the fact that if two vectors are perpendicular, cosine of the angle θ angle between them must be 0. $\cos(\theta) = \frac{\vec{e_1} \cdot \vec{e_2}}{||\vec{e_1}||||\vec{e_2}||} = \frac{\frac{-\sqrt{3}}{4} + \frac{\sqrt{3}}{4}}{||\vec{e_1}||||\vec{e_2}||} = 0$.

Now, the scalars a and b are determined by projecting $\vec{v}$ onto each of the basis vectors $\vec{e_1}$ and $\vec{e_2}$. We showed that both $\vec{e_1}$ and $\vec{e_2}$ are unit vectors, so have length 1. This simplifies the vector projection formula:

$$a = \frac{\vec{v} \cdot \vec{e_1}}{||\vec{e_1}||^2} = \vec{v} \cdot \vec{e_1} = (4,5) \cdot (\sqrt{3}/2, 1/2) = 2\sqrt{3} + 5/2$$

$$b = \vec{v} \cdot \vec{e_2} = (4,5) \cdot (-1/2, \sqrt{3}/2) = -2 + \frac{5\sqrt{3}}{2}$$

5. Let $\vec{u} = (4,1)$ and $\vec{v} = (2,3)$. Calculate $\vec{u} \times \vec{v}$. Use your result to find the angle θ between $\vec{u}$ and $\vec{v}$.

SOLUTION.

For the purposes of taking the cross product, let's write $\vec{u} = (4,1,0)$ and $\vec{v} = (2,3,0)$. $\vec{u} \times \vec{v} = \mathbf{i} \det \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} - \mathbf{j} \det \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix} + \mathbf{k} \det \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} = 10\mathbf{k} = (0,0,10)$. Recall that $||\vec{u} \times \vec{v}|| = ||\vec{u}||||\vec{v}|| \sin\theta$ where θ is the angle between the two vectors. $||\vec{u}|| = \sqrt{17}$ and $||\vec{v}|| = \sqrt{13}$. Plugging in these values, we obtain $\sin\theta = \frac{10}{\sqrt{17}\sqrt{13}}$, which implies $\theta = \arcsin(\frac{10}{\sqrt{221}}) \approx 42.27°$.

6. Find the work done by the force $\vec{F} = 6\mathbf{i} + 8\mathbf{j}$ pounds in moving an object from $(1,0)$ to $(6,8)$ where distance is in feet.

SOLUTION.

The formula for work done is Work = **Force** · **Displacement**. The displacement vector is from (1, 0) to (6, 8), or $(5,8)$.

Work $= (6,8) \cdot (5,8) = 30 + 64 = 94$ lb-ft.

7. Find how much work you would do against the force of gravity ($\vec{F} = -mg\mathbf{j}$) to move an object of mass 5 kg from $(0,0)$ to $(0,\sqrt{2})$, in units of meters. Do the same in moving it from $(0,0)$ to $(1,1)$, and compare your answer. How much work would you do in moving it from $(0,0)$ to $(8,0)$?

SOLUTION.

Since the mass of the object is 5 kg, $\vec{F} = -(5)(9.8)\mathbf{j} = -49\mathbf{j}$. The displacement vector from (0,0) to $(0,\sqrt{2})$ is $\vec{D_1} = 0\mathbf{i} + \sqrt{2}\mathbf{j}$. So the work done by the force of gravity to move the object is $(0,-49) \cdot (0,\sqrt{2}) = 0 - 49\sqrt{2} = -49\sqrt{2}$ Joules, so the work done by you against the force of gravity is $49\sqrt{2}$ Joules. The displacement vector from (0,0) to (1,1) is $\vec{D_2} = 1\mathbf{i} + 1\mathbf{j}$. So the work required in this case is $(0,-49) \cdot (1,1) = -49$

Joules, or 49 Joules done by you. In both of these cases, the work seemed to be 49 multiplied by the **j** component of the displacement vector.

Based on this, we might conjecture that the work done in moving the object from (0,0) to (8,0) will be (-49)(0) = 0. Let's compute to see if we are right. The displacement vector is $\vec{D_3} = 8\mathbf{i} + 0\mathbf{j}$. The work required is therefore $(0, -49) \cdot (8, 0) = 0 + 0 = 0$.

8. Given three points: $\vec{A} = (0, 5, 3), \vec{B} = (2, 7, 0), \vec{C} = (-5, -3, 7)$

   (a) Which point is closest to the xz-plane? Explain your reasoning.

   (b) Which point lies on the xy-plane? Explain your reasoning.

   SOLUTION.

   a) The distance of a point from the xz-plane is simply the absolute value of the y-coordinate. You should try to draw a picture to visualize this. Thus, $\vec{C}$ is the closest with a y-value of -3.

   b) A point is on the xy-plane if its z-coordinate is 0. Thus, point $\vec{B}$ is the point we're looking for.

9. Determine the equation of the plane spanned by the vectors:

$$\vec{u} = 1\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$$
$$\vec{v} = 2\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}$$

   and which contains the origin.

   SOLUTION.

   Since both $\vec{u}$ and $\vec{v}$ lie in the plane and are in different directions, we may take their cross product to find the Normal vector to the plane and determine its equation:

$$\vec{u} \times \vec{v} = (12\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}) - (6\mathbf{k} - 12\mathbf{i} + 4\mathbf{j}) = 24\mathbf{i} - 8\mathbf{j}$$

   Thus the equation for the plane is $24x - 8y = C$. The plane contains the origin, so $24(0) - 8(0) = 0 = C$. So the solution is $24x - 8y = 0$.

10. Find the curvature of the line parameterized by $\vec{r}(t) = (1, 1, 1) + (2, 3, 4)t$.

   SOLUTION.

   $\vec{r}(t) = (1 + 2t, 1 + 3t, 1 + 4t)$
   $\vec{r}'(t) = (2, 3, 4)$

$\vec{T}(t) = \vec{r'}(t)/\sqrt{4+9+16} = 1/\sqrt{29}(2,3,4)$
$\vec{T'}(t) = 0$
$\kappa = ||\vec{T'}(t)||/||\vec{r'}(t)|| = 0/\sqrt{29} = 0$

NOTE: Straight lines always have 0 curvature!

11. Find the arc length of the helix

$$\vec{r}(t) = a\sin(t)\mathbf{i} + a\cos(t)\mathbf{j} + ct\mathbf{k}$$

for $0 \leq t \leq 2\pi$.

SOLUTION.

$\vec{r'}(t) = a\cos(t)\mathbf{i} - a\sin(t)\mathbf{j} + c\mathbf{k}$

Arc Length $= \int_0^{2\pi} ||\vec{r'}(t)||dt = \int_0^{2\pi} \sqrt{a^2cos^2(t) + a^2sin^2(t) + c^2}dt$

$= \int_0^{2\pi} \sqrt{a^2+c^2}dt = 2\pi\sqrt{a^2+c^2}$

12. Find the equation of the plane orthogonal to the curve

$$\vec{r}(t) = (8t^2 - 4t + 3)\mathbf{i} + (\sin(t) - 4t)\mathbf{j} - \cos(t)\mathbf{k}$$

at the point $t = \pi/3$.

SOLUTION.

The Normal vector for the plane will be the tangent vector to the curve at $t = \pi/3$. We will also need to know $r(\pi/3)$.

$$\vec{r}(\pi/3) = (8\pi^2/9 - 4\pi/3 + 3, \sqrt{3}/2 - 4\pi/3, -1/2)$$
$$\vec{r'}(t) = (16t - 4)\mathbf{i} + (\cos(t) - 4)\mathbf{j} + \sin(t)\mathbf{k}$$
$$\vec{r'}(\pi/3) = (16\pi/3 - 4, 1/2 - 4, \sqrt{3}/2)$$

Thus the equation for the plane is

$$(16\pi/3 - 4)x + (-7/2)y + (\sqrt{3}/2)z = C$$

Using the point from above,

$$(16\pi/3 - 4)(8\pi^2/9 - 4\pi/3 + 3) + (-7/2)(\sqrt{3}/2 - 4\pi/3) + (\sqrt{3}/2)(-1/2) = C$$

So the equation for the plane is

$(16\pi/3 - 4)x + (-7/2)y + (\sqrt{3}/2)z =$
$(16\pi/3 - 4)(8\pi^2/9 - 4\pi/3 + 3) + (-7/2)(\sqrt{3}/2 - 4\pi/3) + (\sqrt{3}/2)(-1/2)$

13. Determine the curvature κ of the helical curve parametrized by:

$$\vec{r}(t) = 7\sin(3t)\mathbf{i} + 7\cos(3t)\mathbf{j} + 14t\mathbf{k}$$

at $t = \pi/9$.

SOLUTION.

$\vec{r'}(t) = 21\cos(3t)\mathbf{i} - 21\sin(3t)\mathbf{j} + 14\mathbf{k}$

$||\vec{r'}(t)|| = \sqrt{21^2\cos^2(3t) + 21^2\sin^2(3t) + 14^2} = \sqrt{441 + 196} = \sqrt{637}$

$\vec{T}(t) = 1/\sqrt{637}(21\cos(3t)\mathbf{i} - 21\sin(3t)\mathbf{j} + 14)$

$\vec{T'}(t) = 1/\sqrt{637}(-63\sin(3t)\mathbf{i} - 63\cos(3t)\mathbf{j})$

$||\vec{T'}(t)|| = 1/\sqrt{637}(\sqrt{63^2\sin^2(3t) + 63^2\cos^2(3t)}) = 63/\sqrt{637}$

$\kappa = \frac{||\vec{T'}(t)||}{||\vec{r'}(t)||} = \frac{63}{637}$

Note that the curvature is constant for all t, in particular for $t = \pi/9$.

14. The acceleration of a particle's motion is

$$\vec{a}(t) = -9\cos(3t)\mathbf{i} + -9\sin(3t)\mathbf{j} + 2t\mathbf{k}.$$

The particle has initial velocity $\vec{v}_0 = \mathbf{i} + \mathbf{k}$ and intial position $\vec{x}_0 = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

(a) Determine the velocity function $\vec{v}(t)$.
(b) Determine the position function $\vec{x}(t)$.

SOLUTION.

$$\vec{v}(t) = -3\sin(3t)\mathbf{i} + 3\cos(3t)\mathbf{j} + t^2\mathbf{k} + \vec{C}$$
$$\vec{v}(0) = \mathbf{i} + \mathbf{k} = (0, 3, 0) + \vec{C}$$
$$\vec{C} = (1, -3, 1)$$
$$\vec{v}(t) = (-3\sin(3t) + 1)\mathbf{i} + (3\cos(3t) - 3)\mathbf{j} + (t^2 + 1)\mathbf{k}$$
$$\vec{x}(t) = (\cos(3t) + t)\mathbf{i} + (\sin(3t) - 3t)\mathbf{j} + (t^3/3 + t)\mathbf{k} + \vec{D}$$

$$\vec{x}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k} = (1, 0, 0) + \vec{D}$$

$$\vec{D} = (0, 1, 1)$$

$$\vec{x}(t) = (\cos(3t) + t)\mathbf{i} + (\sin(3t) - 3t + 1)\mathbf{j} + (t^3/3 + t + 1)\mathbf{k}$$

15. Determine the position $\vec{r}(t) = (x(t), y(t))$ of a projectile fired from the point $(8, 3)$ with an initial speed of 20 f/s at an angle of 30°. **Be sure to show all your work**, not just the final formulas.

SOLUTION.

Start with the the fact that the acceleration due to gravity is $\vec{a}(t) = -32\,\mathbf{j}$, where $\mathbf{j} = (0, 1)$, then integrate Newton's second law twice. Velocity is $\vec{v}(t) = \int \vec{a}(t)dt = \int -32dt\mathbf{j} = -32t\mathbf{j} + \vec{C_1}$. To find this first constant, consider the initial condition $\vec{v}(0) = 20\cos(30°)\mathbf{i} + 20\sin(30°)\mathbf{j} = 10\sqrt{3}\mathbf{i} + 10\mathbf{j}$. $\vec{v}(0) = -32(0)\mathbf{j} + \vec{C_1} = \vec{C_1} = 10\sqrt{3}\mathbf{i} + 10\mathbf{j}$. Thus, $\vec{v}(t) = -32t\mathbf{j} + 10\sqrt{3}\mathbf{i} + 10\mathbf{j}$.

The position of the projectile is $\vec{r}(t) = \int \vec{v}(t)dt = \int(-32t\mathbf{j} + 10\sqrt{3}\mathbf{i} + 10\mathbf{j})dt = -16t^2\mathbf{j} + 10\sqrt{3}t\mathbf{i} + 10t\mathbf{j} + \vec{C_2}$. To find the second constant, consider the initial condition $\vec{r}(0) = (8, 3) = 8\mathbf{i} + 3\mathbf{j}$. $\vec{r}(0) = \vec{C_2} = 8\mathbf{i} + 3\mathbf{j}$. Then $\vec{r}(t) = (10\sqrt{3}t + 8)\mathbf{i} + (-16t^2 + 10t + 3)\mathbf{j} = (10\sqrt{3}t + 8, -16t^2 + 10t + 3)$.