Chapter 1

What is calculus about?

⊕ **Calculus** *provides a framework to ask and answer quantitative questions about our world. It is the language used to formulate most scientific laws, and serves as the standardized platform upon which much of technology is built. Indeed, calculus is the* **operating system** *of science and engineering.*

1.1 The two central problems of calculus

Rate of change: zooming in locally. We see change and motion in the world around us every day. For example, driving down a straight road in one direction changes the car's position, as measured by the *odometer*. The *speedometer* measures how fast we're going, or the rate at which the position changes. Suppose our car is equipped with a very precise odometer recording how many feet F the car has traveled after t seconds. The graphs of $F(t)$ for two trips are shown in Figure 1.1. In (a), the car moves uniformly at a constant speed. At $t = 0$ the odometer is set to $F = 0$, and $F(t) = 88t$ is linear. In (b), the car is at rest at $t = 0$ when the accelerator is pressed down, and $F(t) = 16t^2$ is quadratic.

⊕ *If the car's speedometer is broken, how would you use the odometer output* $F(t)$ *to find your speed?*

We already know how to solve this problem for the linear case in (a). The familiar formula for uniform motion, *distance* = *velocity* × *time*, says that the distance $\Delta F = F(t_1) - F(t_0)$ traveled during the time interval $[t_0, t_1]$ seconds is

$$\Delta F \; = \; v \, \Delta t, \tag{1.1}$$

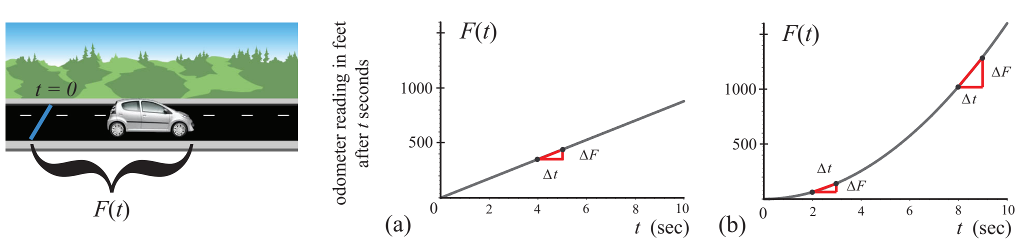

Figure 1.1: (a) The odometer reading with time for a car traveling with constant speed. (b) The odometer reading for a car traveling with constant acceleration, starting from rest.

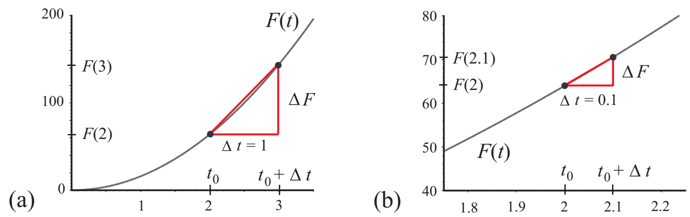

Figure 1.2: (a) The slope of the red hypoteneuse gives the rate of change of $F(t)$ over $[2, 3]$, and a rough estimate for the velocity at $t = 2$ seconds. (b) Over the interval $[2, 2.1]$, the parabolic graph of $F(t)$ is barely distinguishable from the line segment. The slope of this segment provides a closer approximation to the instantaneous velocity at $t = 2$ seconds.

where v is the constant velocity[1], and $\Delta t = t_1 - t_0$ is the length of the time interval. Solving (1.1) for v yields $v = \Delta F/\Delta t$. For the choice of $t_0 = 4$ and $t_1 = 5$ as in Figure 1.1 (a),

$$v = \frac{\Delta F}{\Delta t} = \frac{F(t_1) - F(t_0)}{t_1 - t_0} = \frac{F(5) - F(4)}{5 - 4} = \frac{440 - 352}{1} = 88 \text{ feet/second.} \tag{1.2}$$

We recognize the velocity v in (1.2) as the slope of the line in (a), or the steepness of the linear graph. Any two points on the line give the same slope or velocity.

In Figure 1.1 (b), the velocity is no longer constant, but increases as the car accelerates. Suppose we want to know the **instantaneous velocity** *at* $t_0 = 2$ seconds. Over a short interval around $t_0 = 2$, as in Figure 1.2 (a), the graph of $F(t)$ resembles a line segment, and can be approximated there by the hypoteneuse of the red triangle. Its slope

$$\frac{\Delta F}{\Delta t} = \frac{F(3) - F(2)}{3 - 2} = \frac{144 - 64}{1} = 80 \text{ feet/second} \tag{1.3}$$

is the rate of change of $F(t)$ over the interval $[2, 3]$ of length $\Delta t = 1$. It provides an estimate for the velocity at $t_0 = 2$.

To get a better estimate, we zoom in more closely and look at the rate of change of $F(t)$ over a shorter time interval, such as $[2, 2.1]$ with $\Delta t = 0.1$ as in Figure 1.2 (b). Then

$$\frac{\Delta F}{\Delta t} = \frac{F(2.1) - F(2)}{2.1 - 2} = \frac{70.56 - 64}{0.1} = 65.6 \text{ feet/second.} \tag{1.4}$$

On this small scale, a piece of the curve looks like a line segment. It is natural then to speak of the **slope of the curve** at $t_0 = 2$, or the steepness of the graph there. Zooming in even more, for the interval $[2, 2.01]$ with $\Delta t = 0.01$, we obtain $\Delta F/\Delta t = 64.16$ f/s, and $\Delta F/\Delta t = 64.016$ f/s for the interval $[2, 2.001]$ with $\Delta t = 0.001$. We'd probably all agree that if our speedometer were working, it would read precisely 64 feet/second at exactly 2 seconds after the start, and that the slope of the parabolic curve at $t_0 = 2$ is 64.

⊕ *The central problem of* **differential calculus** *is to find the rate of change of a function at a point, which is the slope of the graph there.*

In general, consider a function $y(x)$. If x is increased from x_0 to $x_1 = x_0 + \Delta x$, then $y(x)$ increases (or decreases) from $y(x_0)$ to $y(x_0 + \Delta x)$ by an amount $\Delta y = y(x_0 + \Delta x) - y(x_0)$. The rate of change of $y(x)$ over the interval $[x_0, x_0 + \Delta x]$ is

$$\frac{\Delta y}{\Delta x} = \frac{y(x_0 + \Delta x) - y(x_0)}{\Delta x}\ , \tag{1.5}$$

[1]Velocity can be positive or negative depending on the direction of travel, while speed s is the absolute value of the velocity, $s = |v| \geq 0$. In the cases considered here they are the same.

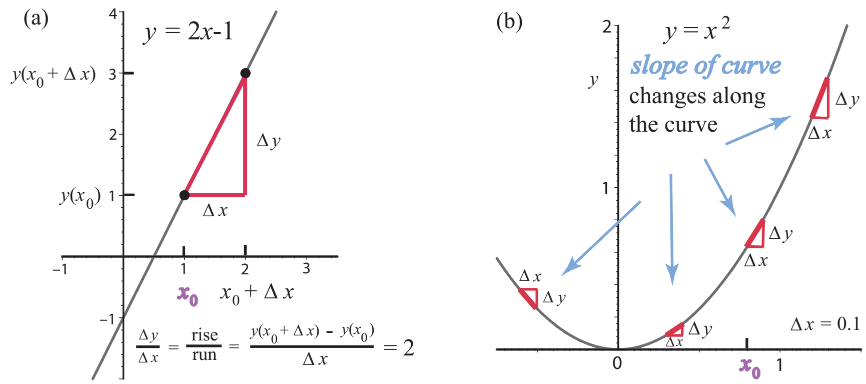

Figure 1.3: (a) The rate of change of $y(x) = 2x - 1$ with respect to x is 2, the slope of the line. (b) The nonlinear function $y(x) = x^2$ has a parabolic graph. The rate of change of y with respect to x at a point $x = x_0$ is the *slope of the curve*, or the steepness there.

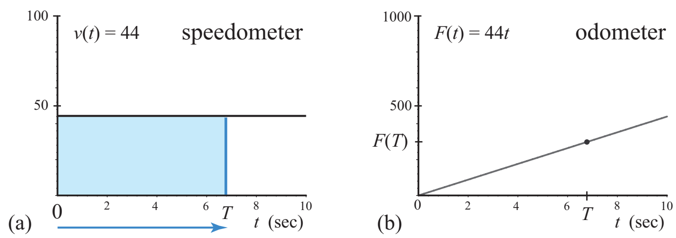

Figure 1.4: The odometer value $F(T)$ in (b) after T seconds is the area of the shaded region under the graph of the speedometer function $v(t)$ in (a) over the interval $[0, T]$.

as illustrated in Figure 1.3 (a) for a line. In differential calculus, we develop mathematics to find the rate of change of a function $y(x)$ **at** a point $x = x_0$. The idea is to examine the ratio in (1.5) as Δx goes to 0 and we zoom in more and more closely around $x = x_0$. For motion, the rate of change of the position is the instantaneous velocity.

Area under a curve: summing up globally. Let's again consider driving a car in one direction down a straight road. Suppose now that our car is equipped with a working speedometer which can record the instantaneous velocity $v(t)$ of the car in feet/second.

⊕ *If the car's odometer is broken, how would you use the speedodometer output $v(t)$ to find how far you've gone?*

For the case of constant velocity v, over the time interval $[0, t]$, Equation (1.1) yields $F(t) - F(0) = v \cdot (t - 0)$. With $F(0) = 0$, we have

$$F(t) = vt, \tag{1.6}$$

obtaining the odometer output $F(t)$ from speedometer data. For example, if the speedometer needle sits on 44 feet/second (30 miles per hour) throughout the trip, then we drive 440 feet, and $F(10) = 44 \cdot 10 = 440$. After T seconds, $F(T) = 44 \cdot T$, which is the area of the rectangular region under the graph of $v(t) = 44$, as illustrated in Figure 1.4.

If the velocity is not constant, such as when the accelerator is held down, we can still find the odometer reading $F(T)$ after T seconds from the speedometer function $v(t)$, by finding the area under the graph of $v(t)$ over the interval $[0, T]$. The idea is to break up

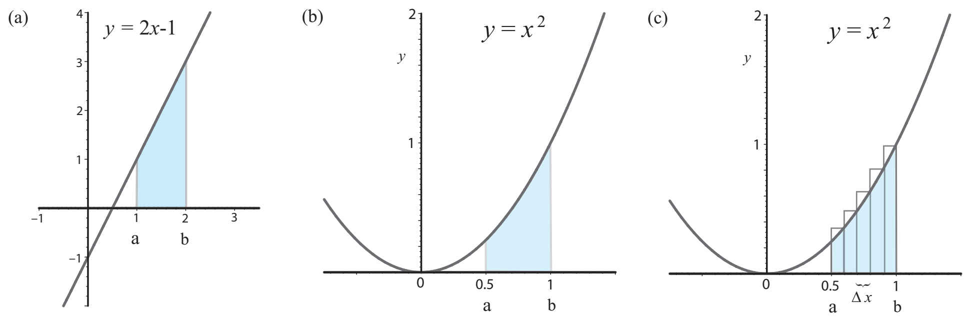

Figure 1.5: (a) The shaded region under the graph of the function $y(x) = 2x - 1$ on $[a, b]$ is a trapezoid with base width $w = b - a$ and heights $h_1 = y(a)$ and $h_2 = y(b)$. Its area is $w \cdot (h_1 + h_2)/2 = 1 \cdot (1 + 3)/2 = 2$. (b) The shaded region under the parabolic graph of $y(x) = x^2$ on $[\frac{1}{2}, 1]$ is *not* a shape whose area we know from basic geometry. (c) The area of the shaded region can be approximated by summing up the areas of the rectangles.

$[0, T]$ into many smaller subintervals, each of length Δt, and to approximate how far we go during each subinterval of time, assuming a constant velocity on each subinterval. Then we sum up the results over the entire interval to obtain an estimate for how far we go during $[0, T]$. The estimate gets better and better as Δt gets smaller and smaller. We'll explore this process in more detail later.

⊕ *The central problem of* **integral calculus** *is to find the area under the graph of a function on an interval.*

In general, consider a function $y(x)$ on an interval $[a, b]$. If the function has a line for a graph, as in Figure 1.5 (a), then we already know how to find the area. However, for a nonlinear function such as $y(x) = x^2$ in Figure 1.5 (b), the region under the graph is a new shape whose area we *don't* know from basic geometry. By breaking up the interval $[a, b]$ into many subintervals of length Δx, we can estimate the area under the curve by the sum of the areas of the boxes, as illustrated in Figure 1.5 (c) for 5 boxes. The approximation gets better and better the more boxes we consider, or the smaller Δx gets. As in differential calculus, the idea is that we obtain the exact result as Δx goes to 0.

⊕ *The two great problems of calculus may seem unrelated. In fact, they are inextricably intertwined. The odometer and speedometer readings in a car may seem to evolve independently. In fact, one function can be obtained from the other, and vice versa, using techniques we'll develop in calculus. The fundamental relationship exhibited by the odometer and speedometer functions lies at the foundation of calculus, and is repeated throughout mathematics, the sciences, and engineering.*

BRIEF SUMMARY : The two main problems in calculus are finding the rate of change of a function at a point, and finding the area under the graph of a function on an interval.

Exercises

1. Let $F(t) = 66t$ feet be the odometer output after t seconds for a car driving in one direction down a straight road. Find the car's speed in feet/second, first using the pair of points at times $t_0 = 1$ and $t_1 = 2$, then using $t_0 = 4$ and $t_1 = 8$. What is the car's speed in miles per hour, and in meters/second?

2. Let $x(t) = 40t$ be the position of a car on a long straight road as a function of time t, where t is measured in hours, and x is measured in miles. (a) How far has the car gone after 15 minutes, after 1/2 hour, after 45 minutes, after 1 hour? (b) Find the rate of change of x with respect to t using your measurements from (a). Interpret the rate of change of the position with time using more common language.

3. Given a length x, let $f(x) = 2x$ be the function which doubles x. Find the rate of change of $f(x)$ over the interval $[x_0, x_0 + \Delta x]$ for the following values of x_0 and Δx: (a) $x_0 = 0$, $\Delta x = 0.5$ (b) $x_0 = 1$, $\Delta x = 1$ (c) $x_0 = 1$, $\Delta x = 3$ (d) $x_0 = 3$, $\Delta x = 0.1$

4. Let $C(r) = 2\pi r$ be the circumference of a circle of radius r. (a) What is the circumference for $r = 1, 2, 3, 4$? (b) Find the rate of change of C with respect to r, using three different pairs of points: $r_0 = 1,\ r_1 = 2$, $r_0 = 1,\ r_1 = 4$, and $r_0 = 3,\ r_1 = 5$. (c) Find the area under the graph of $C(r)$ on the interval $[0, 1]$. Do the same for $[2, 4]$.

5. Let $y(x) = mx + b$, $m \neq 0$. Find the rate of change of $y(x)$.

6. Consider a square of side x. Let $P(x) = 4x$ be its perimeter and $A(x) = x^2$ its area. (a) Let $x_0 = 1$. If x_0 is increased by $\Delta x = 1$, find the increases ΔP and ΔA in the perimeter and area, respectively. Do the same for $x_0 = 2$, with $\Delta x = 1$ and then $\Delta x = 2$. (b) Find the rate of change of $P(x)$. (c) Find the rate of change of $A(x)$ over $[1, 2]$, $[2, 3]$, and $[2, 4]$.

7. For $F(t) = 88t$ as in Figure 1.2, estimate the instantaneous velocity at $t_0 = 8$ seconds by finding the rate of change of $F(t)$ over the following intervals (a) $[8, 9]$ (b) $[8, 8.1]$ (c) $[8, 8.01]$ (d) $[8, 8.001]$. Convert your estimate to miles per hour. What type of car would likely be required to accelerate to this speed after 8 seconds?

8. For $F(t) = 88t$ as in Figure 1.2, estimate the instantaneous velocity at $t_0 = 8$ seconds by finding the rate of change of $F(t)$ over the following intervals (a) $[7, 9]$ (b) $[7.9, 8.1]$ (c) $[7.99, 8.01]$ (d) $[7.999, 8.001]$. Compare with the previous exercise.

9. Find the rate of change of $y(x) = x^3$ over the following intervals (a) $[0, 1]$ (b) $[1, 2]$ (c) $[2, 3]$ (d) $[1, 1.1]$ (e) $[1, 1.01]$ (f) $[1, 1.001]$. What do you think is the exact answer for the rate of change of $y(x) = x^3$ at $x_0 = 1$?

10. Let $y(x) = 5x + 2$. If x is inceased by Δx from $x = x_0$, how much does y increase?

11. Let $y(x) = 2x$. Find the area of the region under the graph of $y(x)$ on the following intervals (a) $[0, 1]$ (b) $[1, 2]$ (c) $[0, 2]$ (d) $[5, 10]$.

12. Consider the function $y(x) = x^2$, whose graph is shown in Figure 1.3. (a) Find the rate of change of $y(x)$ over the interval $[x_0, x_0 + \Delta x]$ for the following values of x_0 and Δx: (*i*) $x_0 = 0$, $\Delta x = 0.5$ (*ii*) $x_0 = 0$, $\Delta x = 0.1$ (*iii*) $x_0 = 1$, $\Delta x = 1$ (*iv*) $x_0 = 1$, $\Delta x = 0.1$ (*v*) $x_0 = -1$, $\Delta x = 0.1$.

13. Find the rate of change of $y(x) = x^2$ over the interval $[x_0, x_0 + \Delta x]$ for the following values of x_0 and Δx: (a) $x_0 = 0$, $\Delta x = 0.05$ (b) $x_0 = 0$, $\Delta x = 0.001$ (c) $x_0 = 1$, $\Delta x = 0.05$ (d) $x_0 = 1$, $\Delta x = 0.001$ (e) $x_0 = -1$, $\Delta x = 0.05$ (f) $x_0 = -1$, $\Delta x = 0.001$.

14. Let $v(t) = 22$ f/s be the speedometer function for a car driving down a straight road. How far has the car gone after 2 seconds, or after 10 seconds? Do the same for $v(t) = 44t$ f/s.

15. Let $v(t) = t^2$ f/s be the speedometer function for a car driving one way down a straight road. Estimate how far the car has gone after 2 seconds? How about after 10 seconds?

16. In Figure 1.5 (b), use a trapezoid with base from $\frac{1}{2}$ to 1 to estimate the area of the shaded region under $y = x^2$ on $[\frac{1}{2}, 1]$. In (c), find the sum of the areas of the rectangles to estimate this area.

17. Consider a cube of side x. Each of the six faces has area x^2. Let $S(x) = 6x^2$ be the surface area of the cube, and $V(x) = x^3$ be its volume. (a) Plot the graphs of both functions on the same axes. (b) For which side length x is the volume and surface area of the cube equal? Find the rate of change of both functions over the interval $[x_0, x_0 + \Delta x]$ for the following values of x_0 and Δx: (c) $x_0 = 0$, $\Delta x = 0.1$ (d) $x_0 = 0$, $\Delta x = 0.01$ (e) $x_0 = 2$, $\Delta x = 0.1$ (f) $x_0 = 2$, $\Delta x = 0.01$ (g) $x_0 = 10$, $\Delta x = 0.1$ (h) $x_0 = 10$, $\Delta x = 0.01$.

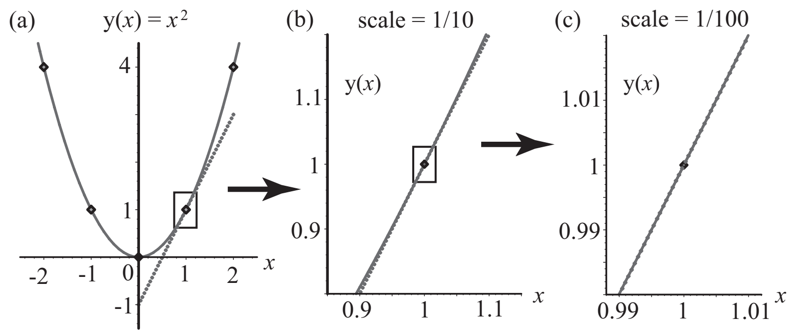

Figure 1.6: Zooming in on the graph of $y(x) = x^2$ around $x_0 = 1$. In (a) the dotted line is tangent to the graph at $x_0 = 1$, intersecting only at the point $(1, 1)$. The slope of the curve at $x_0 = 1$ is the slope of this tangent line. On smaller and smaller scales, as in (b) and (c), the tangent line becomes practically indistinguishable from the curve.

18. Let $V(x) = x^3$ be the volume of a cube of side x. (a) Find the rates of change of $V(x)$ over the intervals $[2, 2.1]$, $[2, 2.01]$, and $[2, 2.001]$ to estimate the rate of change of $V(x)$ **at** $x_0 = 2$. (b) Use 4 rectangles similar to Figure 1.5 (c) to estimate the area under the graph of $V(x)$ on $[0, 2]$, then use 8 rectangles.

1.2 The derivative of a polynomial

Let's focus on the problem of finding the rate of change of a nonlinear function such as $y(x) = x^2$ at a point $x = x_0$. In Figure 1.3 (b) we see that the steepness of the graph depends on location. At the bottom of the valley near $x = 0$ the topography is almost flat, while climbing a slope as steep as that found near $x = 1$ would be challenging.

⊕ *A sufficiently small piece of any smooth curve looks almost like a line segment.*

In Figure 1.6 the graph of $y(x) = x^2$ is examined on smaller and smaller length scales around $x_0 = 1$. The line which intersects the curve only at the point $(1, 1)$, and which has the same steepness as the parabola at $x_0 = 1$, is called the **tangent line**. It captures the trend of the graph at $x_0 = 1$, and closely approximates the function near $x_0 = 1$. Indeed, by the **slope of the curve** at $x_0 = 1$, we mean the slope of this tangent line.

⊕ *The rate of change of $y(x)$ with respect to x at $x = x_0$ is the slope of the line tangent to the graph of $y(x)$ at $x = x_0$.*

Example 1.2.1. Find the rate of change of $y(x) = x^2$ with respect to x at $x_0 = 1$.

Solution: We must find the slope of the line tangent to the parabola at the point $(1, 1)$. However, $(1, 1)$ is the only point we know on this line, and after all, it takes *two* to tango! We would need to know a second point on the tangent line to find its slope directly. To **approximate** this slope, we use a nearby point on the parabola, whose coordinates we do know. The two points $(1 + \Delta x, y(1 + \Delta x))$ and $(1, 1)$ determine the **secant line** shown in Figure 1.7 (a). As Δx approaches 0 in Figure 1.7 (b) and (c), the point $(1+\Delta x, y(1+\Delta x))$ slides down the parabola towards $(1, 1)$, and the secant becomes closer and closer to the tangent. The slope m_{sec} of the secant line is given by

$$m_{sec} = \frac{\Delta y}{\Delta x} = \frac{y(1 + \Delta x) - y(1)}{\Delta x} = \frac{(1 + \Delta x)^2 - 1}{\Delta x} = \frac{2\Delta x + (\Delta x)^2}{\Delta x} = 2 + \Delta x. \qquad (1.7)$$

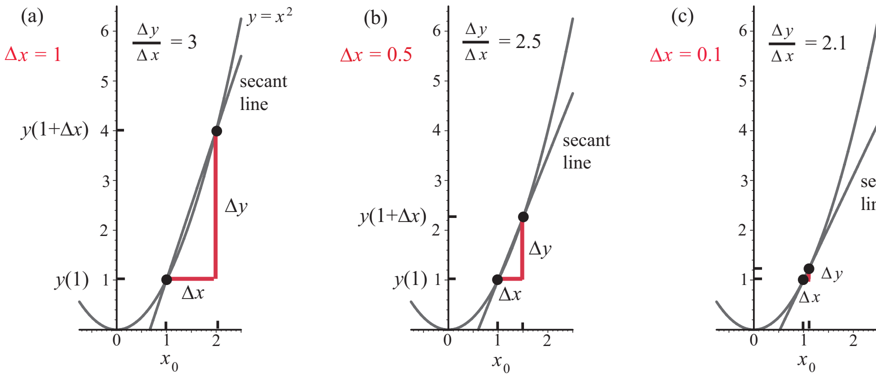

Figure 1.7: The secant line contains the two points on the graph of $y(x) = x^2$ at $x = x_0$ and $x = x_0 + \Delta x$, and *approximates* the tangent line at $x = x_0$. As Δx gets smaller, the approximation gets better, and the secant converges to the tangent as $\Delta x \to 0$.

To obtain the slope m_{tan} of the tangent line, we allow the second point to slide *all the way* down to $(1, 1)$. More precisely, we let Δx approach 0 in Equation (1.7) in a process called *taking the limit* as $\Delta x \to 0$ (detailed in Chapter 2). Then $2 + \Delta x$ approaches 2, and

$$m_{tan} = \lim_{\Delta x \to 0} m_{sec} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2 + \Delta x) = 2. \tag{1.8}$$

⊕ *The slope m_{tan} of the line tangent to the graph of $y(x) = x^2$ at a point $x = x_0$ gives the rate of change of $y(x)$ at $x = x_0$. The slope of the tangent line is called the* **derivative** *of y with respect to x, and is denoted by*

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\ . \tag{1.9}$$

Example 1.2.2. (a) Find the derivative of the function $y(x) = x^2$ for any x. (b) Find the equation of the tangent line at $x_0 = 1$, as well as at $x_0 = -2$.

Solution: (a)

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}\ . \tag{1.10}$$

By expanding $(x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$ in the numerator, we obtain

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x = 2x, \tag{1.11}$$

since $(2x + \Delta x) \to 2x$ as $\Delta x \to 0$. (b) At $x_0 = 1$, the slope of the tangent line is $m = \frac{dy}{dx}(x = 1) = 2 \cdot 1 = 2$. Since $(1, 1)$ is on the line, its $y-$intercept b satisfies $1 = 2 \cdot 1 + b$ or $b = -1$, and the equation is $y = 2x - 1$. At $x_0 = -2$, the slope of the tangent line is $m = \frac{dy}{dx}(x = -2) = 2 \cdot (-2) = -4$. Since $(-2, 4)$ is on the line, its $y-$intercept b satisfies $4 = -4 \cdot (-2) + b$ or $b = -4$, and the equation is $y = -4x - 4$.

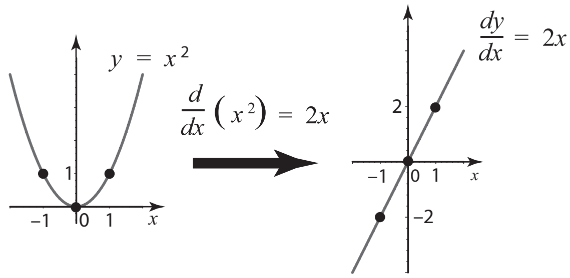

Figure 1.8: The derivative of $y(x) = x^2$ is a new function $dy/dx = 2x$ whose value at x_0 is the slope of the tangent line at $x = x_0$. For example, $\frac{dy}{dx}(-1) = -2$ and $\frac{dy}{dx}(0) = 0$.

Remark: The derivative of $y(x) = x^2$ is a new function $dy/dx = 2x$. The value of dy/dx at $x = x_0$ is the slope of the curve at $x = x_0$. The operation of taking the derivative, where $y = x^2$ is assigned to $dy/dx = 2x$, is illustrated in Figure 1.8. We also write

$$\frac{dy}{dx} = \frac{d}{dx}(y(x)), \qquad \frac{d}{dx}(x^2) = 2x. \tag{1.12}$$

Example 1.2.3. Consider the functions $p(x) = 1$, $q(x) = x$, $f(x) = x^3$, and $g(x) = x^4$. In each case, find the derivative for any x.

Solution: The function $p(x) = 1$ is constant, and its graph is a straight line of zero slope, so that $dp/dx = 0$ for all x. The function $q(x) = x$ is linear and its graph is the straight line with slope 1. Thus $dq/dx = 1$ for all x. To find df/dx, we do the same calculation as in Equation (1.10), except we'll use h instead of Δx, and the cube rather than the square,

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}. \tag{1.13}$$

By expanding $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$ in the numerator, we obtain

$$\frac{df}{dx} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2. \tag{1.14}$$

For $g(x) = x^4$, we need the expansion $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + ...$. Then

$$\frac{dg}{dx} = \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h} = \lim_{h \to 0} 4x^3 + 6x^2h + ... = 4x^3. \tag{1.15}$$

Summarizing our results,

$$\frac{d}{dx}(x^1) = 1, \quad \frac{d}{dx}(x^2) = 2x, \quad \frac{d}{dx}(x^3) = 3x^2, \quad \frac{d}{dx}(x^4) = 4x^3. \tag{1.16}$$

In view of the formulas in (1.16), it is reasonable to expect (and true) that

$$\frac{d}{dx}(x^5) = 5x^4, \quad \frac{d}{dx}(x^6) = 6x^5, \quad ..., \quad \frac{d}{dx}(x^n) = nx^{n-1}, \quad ... . \tag{1.17}$$

⊕ *The derivative of* $y(x) = x^n$ *is* $\frac{dy}{dx} = nx^{n-1}$, n=0,1,2,3,... .

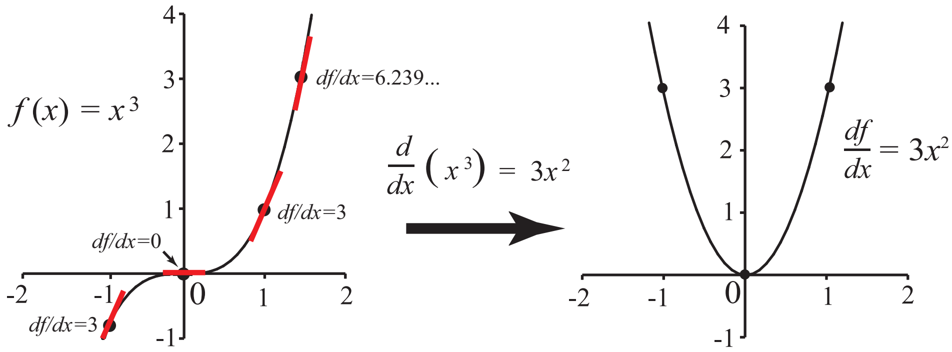

Figure 1.9: The derivative of $f(x) = x^3$ is $df/dx = 3x^2$.

Example 1.2.4. Let $f(x) = x$ and $g(x) = x^2$. Find the following derivatives: (a) $\frac{d}{dx}(2f(x))$, (b) $\frac{d}{dx}(3g(x))$, (c) $\frac{d}{dx}(f+g)$, (d) $\frac{d}{dx}(f-g)$, (e) $\frac{d}{dx}(f+5g)$, (g) $\frac{d}{dx}(x^2-2x+1)$

Solution: (a) $\dfrac{d}{dx}(2x) = \lim\limits_{h\to 0} \dfrac{2(x+h)-2x}{h} = 2\lim\limits_{h\to 0}\dfrac{(x+h)-x}{h} = 2\cdot 1 = 1$, or

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x)), \tag{1.18}$$

with $c = 2$. (b) $\dfrac{d}{dx}(3x^2) = \lim\limits_{h\to 0}\dfrac{3(x+h)^2 - 3x^2}{h} = 3\dfrac{d}{dx}(x^2) = 6x$, or $\dfrac{d}{dx}(3g(x)) = 3\dfrac{dg}{dx}$.

(c) $\dfrac{d}{dx}(x+x^2) = \lim\limits_{h\to 0}\dfrac{(x+h)+(x+h)^2 - x - x^2}{h} = \dfrac{d}{dx}(x) + \dfrac{d}{dx}(x^2) = 1+2x$, or

$$\frac{d}{dx}(f(x)+g(x)) = \frac{df}{dx} + \frac{dg}{dx}\ . \tag{1.19}$$

Similarly, in (d) we have $\frac{d}{dx}(x-x^2) = 1-2x$, or

$$\frac{d}{dx}(f(x)-g(x)) = \frac{df}{dx} - \frac{dg}{dx}\ . \tag{1.20}$$

In (e), $\frac{d}{dx}(f+5g) = \frac{d}{dx}f + 5\frac{d}{dx}g = 1+10x$, and in (f), $\frac{d}{dx}(x^2-2x+1) = 2x-2+0 = 2x-2$.

⊕ *The derivative of a constant times a function is the constant times the derivative; the derivative of a sum is the sum of the derivatives; the derivative of a difference is the difference of the derivatives.*

The properties of the derivative in Equations (1.18), (1.19) and (1.20) are true in general, and allow us to find the derivative of *any* polynomial. For example, the derivative of any quadratic $p(x)$ or cubic $q(x)$ is

$$\frac{dp}{dx} = \frac{d}{dx}(a_2x^2 + a_1x + a_0) = a_2\frac{d}{dx}(x^2) + a_1\frac{d}{dx}(x) + a_0\frac{d}{dx}(1) = 2a_2x + a_1, \tag{1.21}$$

$$\frac{dq}{dx} = \frac{d}{dx}(a_3x^3 + a_2x^2 + a_1x + a_0) = 3a_3x^2 + 2a_2x + a_1. \tag{1.22}$$

Our results are summarized in the following.

Theorem 1.1 (Derivative of any polynomial). $\dfrac{d}{dx}(x^n) = nx^{n-1},\ n = 0, 1, 2, 3, ...,$ *and for any* n^{th} *order polynomial* $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$,

$$\frac{dp}{dx} = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ ... \ + 2a_2 x + a_1. \tag{1.23}$$

Example 1.2.5. Consider

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}\ , \tag{1.24}$$

where $3! = 3 \cdot 2 \cdot 1 = 6$, or 3 *factorial*, and $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. Find df/dx.

Solution:

$$\frac{df}{dx} = \frac{d}{dx}(x) - \frac{1}{3!}\frac{d}{dx}(x^3) + \frac{1}{5!}\frac{d}{dx}(x^5) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\ . \tag{1.25}$$

Example 1.2.6. Find a function $F(x)$ whose derivative is $f(x) = x$. That is, find $F(x)$, such that $dF/dx = x$.

Solution: From (1.16) we have $\dfrac{d}{dx}\left(x^2\right) = 2x$. Dividing both sides by 2 yields $\frac{d}{dx}(\frac{x^2}{2}) = x$, so that $F(x) = x^2/2$ is such a function. It is useful to observe that for any constant C, $\frac{d}{dx}(\frac{x^2}{2} + C) = x + 0 = x$. Then $F(x) = x^2/2 + C$ is an **antiderivative** of $f(x) = x$, since its derivative is $f(x)$.

BRIEF SUMMARY: The rate of change of a function $f(x)$ at a point $x = x_0$, is the slope of the tangent line there, and is called the **derivative**, df/dx. It is found via

$$\frac{df}{dx}(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}\ . \tag{1.26}$$

The key result is that the derivative of x^n is $nx^{n-1}, n = 0, 1, 2, ...,$ which allows us to find the derivative of any polynomial.

Exercises

1. Let $\phi(x) = \frac{1}{3}x$. Plot the graph of $\phi(x)$. Find $[\phi(x + \Delta x) - \phi(x)]/\Delta x$ for (a) $x = 5$ with $\Delta x = 1/2,\ 1/10,\ 1/100$ and (b) $x = 1$ and $\Delta x = 1/2,\ 1/10,\ 1/100$.
2. Let $f(x) = x^2 - 1$. Plot the graph of $f(x)$. Find $[f(x + \Delta x) - f(x)]/\Delta x$ for (a) $x = 1$ and $\Delta x = 1/2,\ 1/10,\ 1/100$ and (b) $x = 0$ and $\Delta x = 1/2,\ 1/10,\ 1/100$.
3. Let $f(x) = \frac{1}{2}x^3$. Plot the graph of $f(x)$. Find $[f(x + \Delta x) - f(x)]/\Delta x$ for (a) $x = 1$ and $\Delta x = 1/2,\ 1/10,\ 1/100$ and (b) $x = 0$ and $\Delta x = 1/2,\ 1/10,\ 1/100$.
4. For each of the following functions, plot it on graph paper along with the tangent line at each of the given points. Estimate the slope of the tangent line at each point and compare your results with Theorem 1.1. (a) $f(x) = \frac{1}{2}x^2 - 1$ at $x_0 = -1, 0, 1$, (b) $f(x) = \frac{1}{4}x^3$ at $x_0 = -1, -1/2, 0, 1/2, 1, 2$, (c) $f(x) = x(1 - x)$ at $x_0 = 0, 1/4, 1/2, 3/4, 1$.
5. For each of the following functions $f(x)$, find the rate of change of $f(x)$ over the interval $[x_0, x_0 + \Delta x]$ for given x_0 and Δx, and sketch the graph of $f(x)$ along with the secant line corresponding to the interval. In each case compare your results with what you obtain for the derivative of $f(x)$ at $x = x_0$ using Theorem 1.1. (a) $f(x) = x^2$ with $x_0 = \frac{1}{2}$ and $\Delta x = 1, 1/2, 1/10, 1/100$, (b) $f(x) = 2x^3 - 1$ with $x_0 = 1$ and $\Delta x = 1, 1/2, 1/10, 1/100$, (c) $f(x) = \frac{1}{2}x^2 - 1$ with $x_0 = -1$ and $\Delta x = 1, 1/2, 1/10, 1/100$.

6. Find the derivative of $f(x)$ at any x_0 for each of the following functions by directly computing the limit in (1.26). In each case find the equation of the tangent line at $x_0 = 1$. (a) $f(x) = \frac{1}{5}x^2$ (b) $f(x) = 3x^2 - 2$ (c) $f(x) = 3x - 4$ (d) $f(x) = x^2 + \pi$ (e) $f(x) = 2x^3 + 4$ (f) $f(x) = x^2 - 2x + 5$ (g) $f(x) = 2x^3 + x$ (h) $f(x) = x^4$
7. Find the derivative of $f(x)$ at any x for the following functions by directly computing the limit $\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. (a) $f(x) = x^2$ (b) $f(x) = 5x^2$ (c) $f(x) = 3x-4$ (d) $f(x) = x^2+2$ (e) $f(x) = 3x^2 + 4$ (f) $f(x) = x^2 - 2x + \pi$ (g) $f(x) = 2x^3 + x$ (h) $f(x) = x^4$
8. Suppose the $f(x)$ and $g(x)$ are functions such that $df/dx = 3x^2$ and $dg/dx = x^3 - 4$. Find (a) $\frac{d}{dx}(f(x) + g(x))$, (b) $\frac{d}{dx}(f(x) - 2g(x))$, (c) $\frac{d}{dx}(4f(x))$, (d) $\frac{d}{dx}(2f(x) - 3)$
9. Find the derivative $\frac{df}{dx}$ for each of the following functions: (a) $f(x) = x^3 - x^2 + 3x - 8$, (b) $f(x) = \pi x^2 + 6x - 3$, (c) $f(x) = x^{100} + 49x^{50} + 1$, (d) $f(x) = 3x^5 - 2x^4 + 7x^3 + x^2 - 6x + 4$.
10. The graph of the circumference $C(r)$ of a circle as a function of its radius r is a straight line. Plot the graph of $C(r)$ for $r \geq 0$ and find dC/dr.
11. The graph of the area $A(r)$ of a circle as a function of its radius r is a parabola. Plot the graph of $A(r)$ for $r \geq 0$ and find its derivative for any $r \geq 0$. Plot the graph of its derivative dA/dr, and compare with your result in the previous problem.
12. Let $P(x) = 4x$ be the perimeter of a square of side x, and $A(x) = x^2$ be the area of the square. Find dP/dx and dA/dx for $x \geq 0$. Plot the graphs of $P(x)$ and $A(x)$, and their derivatives.
13. Let $V(x) = x^3$ be the volume of a cube with side x, and let $S(x) = 6x^2$ be the surface area of the cube. Find dV/dx and dS/dx for any $x \geq 0$. Plot the graphs of $V(x)$ and $S(x)$, and plot the graphs of their derivatives. Evaluate the derivatives at $x_0 = 0, 2, 10$ and compare your results with those from Problem 17 in Section 1.1.
14. Let $g(x) = 1 - x^2/2! + x^4/4!$. Find dg/dx.
15. Find a function $F(x)$ whose derivative is $f(x) = 2x$, that is, find $F(x)$ such that $\frac{dF}{dx} = f(x) = 2x$. Do the same for $f(x) = 3x^2$, x and x^2.
16. **Farenheit $\rightleftharpoons$ Celsius conversion.** A temperature of 32° Farenheit (F) corresponds to 0° Celsius (C), and a temperature of 212° F corresponds to 100° C. (a) Find the equation of the line for converting temperatures from Farenheit to Celsius, and sketch its graph. Use your formula to convert 75° from F to C. (b) What is the rate of change of degrees Celsius with respect to degrees Farenheit? (c) Find a function $F(C)$ for converting temperatures from Celsius to Farenheit, and sketch its graph. Use your formula to convert 15° from C to F. (d) Find the derivative of $F(C)$? (e) The units of the Kelvin (K) scale are the same size as in the Celsius scale, but zero is shifted so that 0° K corresponds to −273.15° C, called *absolute zero*, and 273.15° K corresponds to 0° C. Find the function $K(C)$ converting from C to K, and vice versa, and use your formulas to convert 25° C to K, and 0° K to F. Find dK/dC.
17. **Thermal expansion.** A steel girder is 15 meters long when it is at a temperature of −20° C. When in use the girder will be subjected to temperatures between −20° C and 40° C. As the temperature changes, the steel expands or contracts, so that the length of the girder changes. When the girder is at temperature T its length L is given by $L = L(T) = 15.0036 + 0.00018T$, $-20 \leq T \leq 40$. Graph this equation and find dL/dT, which measures the rate of thermal expansion per degree of temperature change.
18. **Currency exhange.** If the rate of exchange from US Dollars to Euros is 1.27 Dollars/Euro, and the exhange fee is \$10.00, find a formula for converting x Dollars to y Euros, taking into account the exchange fee (which shifts the $y-$intercept). Sketch a graph of this function and find out how many Euros \$500 will buy.
19. **Pressure under water.** The pressure p experienced by a diver at a depth z meters below the surface is given by an equation of the form $p = kz + 1$, where k is a constant and the pressure at the surface $z = 0$ is 1 atmosphere. If the pressure at 100 meters is 10.94 atmospheres, determine k and find the pressure at $z = 50$ meters. What is the meaning of k, and what units are used to measure k? Find dp/dz.

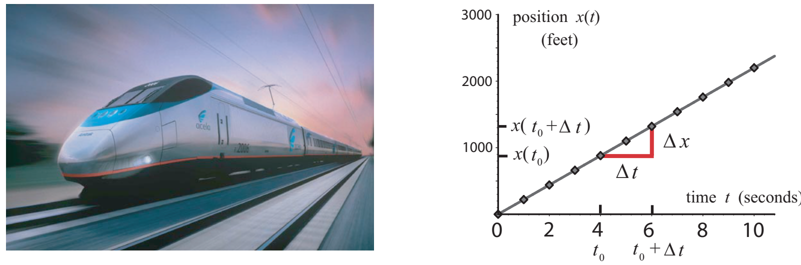

Figure 1.10: High-speed train in uniform motion. Displayed on the right is data on its position x in feet *vs.* time t in seconds, taken at 1 second intervals.

1.3 Instantaneous velocity

Let's again consider motion, which is one of the most important applications of the derivative. The position $x(t)$ of a train speeding uniformly down a track is shown in Figure 1.10. From Equation (1.1), the constant velocity of the train is the slope of the line,

$$v = \frac{\Delta x}{\Delta t}\ , \tag{1.27}$$

measured from any two points t_0 and $t_0 + \Delta t$. For the two data points $(4, 880)$ and $(6, 1320)$ shown, the train travels a distance $\Delta x = x(t_0 + \Delta t) - x(t_0) = 1320 - 880 = 440$ feet in $\Delta t = 6 - 4 = 2$ second. Thus $v = \frac{\Delta x}{\Delta t} = 220$ feet/second, or 150 miles per hour, and the position of the train is given by the linear function $x(t) = v\, t = 220\, t$.

⊕ *For an object in uniform motion, the graph of its position $x(t)$ is a straight line. The velocity is the rate of change of $x(t)$, or the slope of the line.*

Now consider non−uniform motion in one dimension, such as an object falling in earth's gravitational field. After a pebble has been dropped, its downward velocity keeps increasing, to a point, similar to the speedometer reading on a car when you press the accelerator and hold it there. Data on the position $x(t)$ of a falling pebble are shown in Figure 1.11 (a), along with the function $x(t) = -16t^2$ giving the position of the pebble for any time t. As in the previous section, we see in Figure 1.11 (b)−(e) that the closer we zoom in on the graph of $x(t)$ around $t_0 = 1$ second, the more linear it appears, like uniform motion in Figure 1.10.

⊕ *Non−uniform motion looks almost uniform on sufficiently small time scales.*

On small time scales, the trend of $x(t)$ around $t_0 = 1$ is captured closely by the tangent line to its graph at $t_0 = 1$, which is particularly apparent in Figure 1.11 (e).

⊕ *The velocity $v(t)$ of an object at any instant $t = t_0$ is called the* **instantaneous velocity***, and is the slope of the line tangent to the graph of $x(t)$ at $t = t_0$.*

Example 1.3.1. Find the instantaneous velocity $v(t)$ of a falling pebble 1 second after being dropped, and then for any time t.

Solution: To find the instantaneous velocity $v(1)$, we must calculate the slope of the tangent line to $x(t) = -16t^2$ at $t_0 = 1$, or the *derivative* of $x(t) = -16t^2$ at $t_0 = 1$. Fortunately, the

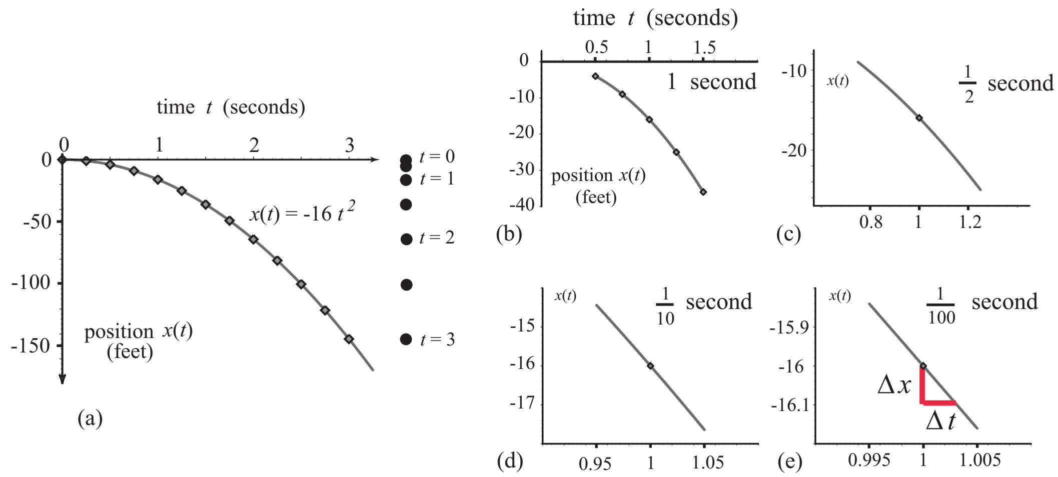

Figure 1.11: (a) Data on the position of a falling pebble at time intervals of 0.25 seconds. The function $x(t) = -16t^2$ gives the position for any time t, neglecting air friction. (b)−(e) Zooming in on the graph of $x(t)$ around $t = 1$ shows that $x(t)$ looks more and more linear when viewed on smaller and smaller time scales, like uniform motion in Figure 1.10.

same procedure we developed in Equations (1.7) and (1.8) still works. The tangent line at $t_0 = 1$ can be approximated with a secant line whose slope is

$$m_{sec} = \frac{\Delta x}{\Delta t} = \frac{x(1+\Delta t) - x(1)}{\Delta t} \,, \tag{1.28}$$

which measures the rate of change of the position $x(t)$ over the time interval $[1, 1 + \Delta t]$. The slope of the tangent line, or instantaneous velocity, is obtained in the limit as $\Delta t \to 0$,

$$v(1) = \frac{dx}{dt}(1) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(1+\Delta t) - x(1)}{\Delta t} \,. \tag{1.29}$$

With $x(t) = -16t^2$, we have

$$v(1) = \lim_{\Delta t \to 0} \frac{-16(1+\Delta t)^2 + 16}{\Delta t} = \lim_{\Delta t \to 0} -32 - 16\Delta t = -32 \text{ f/s}. \tag{1.30}$$

Repeating the same calculation for any t yields

$$v(t) = \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0} -32t - 16\Delta t = -32t \text{ f/s}. \tag{1.31}$$

⊕ *The instantaneous velocity $v(t)$ is the derivative of the position $x(t)$.*

A useful tool for analyzing non−uniform motion is the **average velocity**. Suppose it takes 5 hours including stops to drive from Washington, D.C. to New York City, about 250 miles. While the speedometer reading varies quite a bit, the average speed over the time interval $[0, 5]$ hours is 50 miles per hour since overall we go 250 miles in 5 hours. For the falling pebble, its *average velocity* over the time interval $[t, t + \Delta t]$ is

$$v_{avg}[t, t+\Delta t] = \frac{\Delta x}{\Delta t} = \frac{x(t+\Delta t) - x(t)}{\Delta t} \,. \tag{1.32}$$

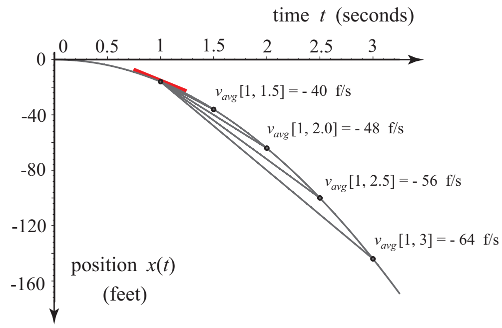

Figure 1.12: The average velocity over each time interval is the slope of the *secant line*. As the interval becomes smaller and smaller, the secant lines approach the red tangent line.

Comparing with Equation (1.31), we see that the instantaneous velocity $v(t)$ is the limit of the average velocity measured over smaller and smaller time intervals,

$$v(t) = \lim_{\Delta t \to 0} v_{avg}[t, t + \Delta t] = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}\ . \tag{1.33}$$

In Table 1 we show the average velocity of the pebble over shorter and shorter time intervals starting at $t = 1$. As $\Delta t \to 0$, $v_{avg}[1, 1 + \Delta t] \to -32$, consistent with (1.30). In Figure 1.12, $v_{avg}[1, 1 + \Delta t]$ is the slope of the secant line containing $t = 1$ and $t = 1 + \Delta t$.

time interval $[1, 1 + \Delta t]$	Δt	$v_{avg}[1, 1 + \Delta t]$	relative error $\mathcal{E}$
$[1, 3]$	2	-64	1
$[1, 2]$	1	-48	1/2
$[1, 1.5]$	1/2	-40	1/4
$[1, 1.25]$	1/4	-36	1/8
$[1, 1.125]$	1/8	-34	1/16
$[1, 1.0625]$	1/16	-33	1/32
$[1, 1.03125]$	1/32	-32.5	1/64
$[1, 1.015625]$	1/64	-32.25	1/128
$[1, 1.0078125]$	1/128	-32.125	1/256
$[1, 1.00390625]$	1/256	-32.0625	1/512

Table 1: Average velocities $v_{avg}[1, 1 + \Delta t]$ of a falling pebble during smaller and smaller time intervals $[1, 1 + \Delta t]$. For each average velocity, the relative error $\mathcal{E} = (v_{avg}[1, 1 + \Delta t] - (-32))/(-32)$ between $v_{avg}[1, 1 + \Delta t]$ and the limit -32 is also shown.

⊕ *The average velocity v_{avg} over a time interval $[t_0, t_0 + \Delta t]$ is the slope of the secant line to $x(t)$ containing the points $t = t_0$ and $t = t_0 + \Delta t$.*

Example 1.3.2. Let the position of a falling particle after t seconds be given by $x(t) = -16t^2$ feet. Find the average velocity in feet/second of the particle over each of the following time intervals: $[0, 1]$, $[1, 2]$, $[2, 3]$.

Solution: For these intervals, we have $v_{avg}[0, 1] = \frac{x(1)-x(0)}{1-0} = \frac{-16-0}{1-0} = -16$ f/s, $v_{avg}[1, 2] = \frac{x(2)-x(1)}{2-1} = \frac{-64-(-16)}{1} = -48$ f/s, and $v_{avg}[2, 3] = \frac{x(3)-x(2)}{3-2} = -80$ f/s.

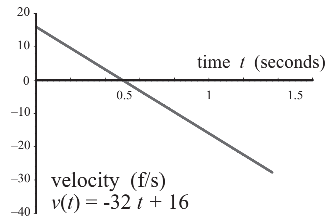

Figure 1.13: The instantaneous velocity of a ball thrown upward with an initial velocity of 16 f/s. The maximum height occurs at $t = \frac{1}{2}$ when the velocity is 0.

Example 1.3.3. The position $x(t)$ of a ball thrown upward from an initial height of 8 feet with an initial velocity of 16 feet/second is given by $x(t) = -16t^2 + 16t + 8$. (a) Find the instantaneous velocity $v(t)$ of the ball. (b) Determine when the ball is going up, when it's falling, its maximum height, when it hits the ground, and its speed $|v(t)|$.

Solution: (a) By Theorem 1.1, the instantaneous velocity is

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}(-16t^2 + 16t + 8) = -32t + 16, \qquad (1.34)$$

whose graph is shown in Figure 1.13. (b) The ball goes up when $v > 0$, which occurs for $t < \frac{1}{2}$. The maximum height of the ball is reached at the instant when it stops ascending ($v > 0$) and begins descending ($v < 0$), that is, when $v = 0$, which occurs at $t = \frac{1}{2}$. Then, $x(\frac{1}{2}) = 12$ feet. The ball falls when $v < 0$ for $t > \frac{1}{2}$. It hits the ground when $x(t) = 0$, occuring at $t = \frac{1}{2} + \frac{\sqrt{3}}{2} = 1.36...$, by the quadratic formula. For $t \leq \frac{1}{2}$, $|v(t)| = v(t)$, but when v is negative for $t > \frac{1}{2}$, $|v(t)| = -v(t)$.

BRIEF SUMMARY: Describing and predicting motion is fundamental to science and engineering. If $x(t)$ represents the position of an object, then its instantaneous velocity $v(t)$ is the rate of change or derivative of the position, $v(t) = dx/dt$.

Exercises

1. Consider a car with the position data $(0, 0)$, $(0.25, 16.25)$, $(0.50, 32.5)$, $(0.75, 48.75)$, with units of (hours, miles). Plot this data as in Figure 1.10. Find the velocity and speed of the car and the equation of the line containing the data. Do the same for the data $(0, 50)$, $(0.33, 33.33)$, $(0.67, 16.67)$, $(1, 0)$.
2. A plane flies 2462 miles (3961 km) directly from Los Angeles to New York, which takes 5 hours and 23 minutes including time for maneuvering after take-off and before landing. What is the plane's average speed over the whole trip?
3. Let the position of a particle after t seconds be given by $x(t) = t^3$ centimeters. Find the average velocity of the particle over each of the following time intervals: $[0, 1]$, $[1, 2]$, $[2, 3]$. Do the same for the position functions $y(t) = -16t^2 + 24t$ feet and $z(t) = t - t^3/6$ meters, with t in seconds.
4. Consider a falling pebble with position function $x(t) = -16t^2$. (a) Plot the graph of $x(t)$. (b) On your graph from (a), plot the secant lines containing the points with coordinates $(\frac{1}{2}, x(\frac{1}{2})$ and $\frac{1}{2} + \Delta t, x(\frac{1}{2} + \Delta t))$, with $\Delta t = 0.4, 0.2, 0.1, 0.05, 0.01$. Find the slope of each secant line. (c) Find the average velocities $v_{avg}[\frac{1}{2}, \frac{1}{2} + \Delta t]$ of the falling pebble during the time intervals $[\frac{1}{2}, \frac{1}{2} + \Delta t]$ with $\Delta t = 0.4, 0.2, 0.1, 0.05$, and 0.01. (d) Find $v(\frac{1}{2})$, the instantaneous velocity of a falling pebble at $t = \frac{1}{2}$ seconds. (e) For each average velocity in (c), find the relative error $\mathcal{E} = \left(v_{avg}[\frac{1}{2}, \frac{1}{2} + \Delta t] - v(\frac{1}{2})\right)/v(\frac{1}{2})$. between $v_{avg}[\frac{1}{2}, \frac{1}{2} + \Delta t]$ and your result in (d).

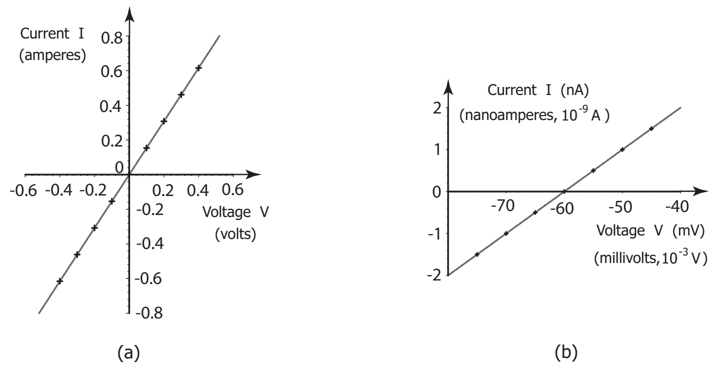

Figure 1.14: In (a), current *vs.* voltage data for a 100 foot copper wire exhibits Ohmic, or linear behavior [7]. Reversing the potential difference between the ends of the wire reverses the current direction. In (b) current *vs.* voltage data for a neuron is plotted.

5. For each of the following position functions, plot it on graph paper along with the tangent line at each of the given points. Estimate the instantaneous velocity at each point by approximating the slope of each tangent line and compare your results with Theorem 1.1. (a) $x(t) = \frac{1}{3}t^2 + 1$ at $t_0 = 0, 1, 2$, (b) $x(t) = \frac{1}{4}t^3$ at $t_0 = 0, 1/2, 1, 2$, (c) $x(t) = 2t(1-t)$ at $x_0 = 0, 1/4, 1/2, 3/4, 1$.

6. For each of the following functions $x(t)$, find the rate of change of $x(t)$ over the interval $[t_0, t_0 + \Delta t]$ for given t_0 and Δt. Sketch the graph of $x(t)$ along with the secant line corresponding to the interval. In each case compare your results with what you obtain for the derivative of $x(t)$ at $t = t_0$ using Theorem 1.1. (a) $x(t) = \frac{1}{2}t^2$ with $t_0 = \frac{3}{2}$ and $\Delta t = 1, 1/2, 1/10, 1/100$, (b) $x(t) = \frac{1}{4}t^3 + t - 1$ with $t_0 = 2$ and $\Delta t = 1, 1/2, 1/10, 1/100$, (c) $x(t) = t - t^2$ with $t_0 = 1$ and $\Delta t = 1, 1/2, 1/10, 1/100$.

7. Find the derivative for each $x(t)$ by directly computing the limit $v(t) = \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$. In each case find the equation of the tangent line at $t_0 = 1$. (a) $x(t) = \frac{1}{3}t^2$ (b) $x(t) = t^3 + 4$ (c) $x(t) = 2t - 3$ (d) $x(t) = t^2 - \pi$ (e) $x(t) = t^4$

8. Let the position of a particle on a line be given by $x(t) = \frac{1}{3}t^3 - 2t^2 + 5$ feet, where t is in seconds. (a) Find the instantaneous velocity $v(t)$ of the particle, (b) Graph $v(t)$ for $t \geq 0$, as well as its speed $|v(t)|$. (c) Find when the object's velocity is 0.

9. For each $x(t)$ find the velocity $v(t) = \frac{dx}{dt}$. (a) $x(t) = -16t^2 + 32t + 6$ f (b) $x(t) = -16t^4 + 64t^2 + 128$ f (c) $x(t) = -32t$ f (d) $x(t) = -5t^2 + 20t$ m (e) $x(t) = t - t^3/6$ m

10. Let $x(t) = t^3 - 3t^2$ be the position of an object. (a) Find $v(t)$. (b) When is the object moving to the right ($v > 0$)? (c) When is it moving to the left ($v < 0$)? (d) Sketch a diagram showing the motion. Repeat the exercise for $x(t) = -t^2 + 6t$.

11. The area of a circle of radius r is $A = \pi r^2$. How fast is the area changing with respect to radius when $r = 1$? How fast is the volume $V = \frac{4}{3}\pi r^3$ of a spherical balloon changing with respect to radius when $r = 2$?

12. If an electrical potential difference V is applied between the ends of a rod, then the resistance R of the rod is the ratio of the applied voltage V to the induced current I, $R = V/I$. The standard unit of resistance is the **ohm** (Ω), equal to 1 volt per ampere ($1\ \Omega = 1\ V/A$). Let us apply a variable potential difference V between the ends of a conductor, and make measurements of the current I as V is varied. Data on the current I *vs.* voltage V for a 100 foot coil of #18 gauge copper wire is shown in Figure 1.14 (a). From this data find the resistance R of the wire. Find the equation of the line in Figure 1.14 (a). What is dI/dV?

13. Every neuron has a thin cloud of positive and negative ions spread over the inner and outer surfaces of the cell membrane, setting up a potential difference across the cell membrane, called the membrane potential. Measuring input and output ionic current pulses through the membrane yields data on the current I *vs.* the membrane potential V [9], as in Figure 1.14 (b). The slope of the line formed by the data is $1/R$, where R is the resistance the neuron. Find R from this data set. Find the equation of the line in Figure 1.14 (b). What is dI/dV?

14. For each velocity function $v(t)$, find a position function $x(t)$ which is an antiderivative of v, that is, $dx/dt = v$. (a) $v(t) = 10t$ (b) $v(t) = -32t$ (c) $v(t) = 22$ (d) $v(t) = 3t^2$ (e) $v(t) = t - t^3$

15. On Earth an object falls with its position given by $x(t) = -4.9t^2$, with x in meters and t in seconds. The corresponding equations on Mars and Jupiter are $x(t) = -1.86t^2$ and $x(t) = -11.44t^2$. On Earth it takes about 1 second for a falling pebble to travel with a speed of 10 m/s. Find how long it takes a pebble to fall with this speed on Mars and Jupiter.

16. An object thrown upward from the surface of the moon with initial velocity v_0 m/s has position $x(t) = -0.8t^2 + v_0 t$. What must v_0 be so that it goes a mile high?

1.4 The integral of a polynomial

The second fundamental problem of calculus concerns the area under the graph of a function on an interval $[a, b]$. If we consider motion in one direction, and the area under a velocity function $v(t) \geq 0$ on $[a, b]$, the idea for a solution becomes apparent. For reasons discussed later, this area will be denoted by

$$\int_a^b v(t)dt. \tag{1.35}$$

We've already seen in Figure 1.4 that for uniform motion, the area under the graph of the constant $v(t) = 44$ f/s over the time interval $[0, 10]$, or $\int_0^{10} 44\, dt$, is the distance traveled during that interval. Via (1.1), this distance is $44 \times 10 = 440$ feet. On the other hand, the distance traveled over $[0, 10]$ is also just the change in position $x(t)$ from $t = 0$ to $t = 10$,

$$\int_0^{10} v(t)\, dt = x(10) - x(0), \tag{1.36}$$

with $x(t) = 44t$ and $x(10) - x(0) = 440 - 0 = 440$ feet. Since the velocity $v(t)$ is the derivative of the position $x(t)$,

$$v(t) = \frac{dx}{dt}\ , \tag{1.37}$$

the position $x(t)$ is an antiderivative of $v(t)$, as discussed in Example 1.2.6. Thus (1.36) suggests that the problem of finding the area under a given function $v(t)$ on an interval $[a, b]$ can be solved by finding an *antiderivative* $x(t)$ of $v(t)$.

The next example shows why, even for non-uniform motion where $v(t)$ is changing, the distance traveled during a time interval $[a, b]$ is the area under the graph of $v(t)$ on $[a, b]$.

Example 1.4.1. The velocity of a cheetah stalking and then suddenly charging an antelope is $v(t) = 10t$ meters/second during the first 2 seconds of the attack. How far does the cheetah run during the time interval $[0, 2]$?

Solution: In the previous section we observed that non$-$uniform motion is approximately uniform on small time scales. It is useful then to break up the time interval $[0, 2]$ into many short subintervals each of duration Δt. In Figure 1.15 (a) the interval $[0, 2]$ is broken up into four subintervals $[t_0, t_1], [t_1, t_2], [t_2, t_3], [t_3, t_4]$, with $\Delta t = \frac{1}{2}$ second, $t_0 = 0,\ t_1 = \frac{1}{2},\ t_2 =$

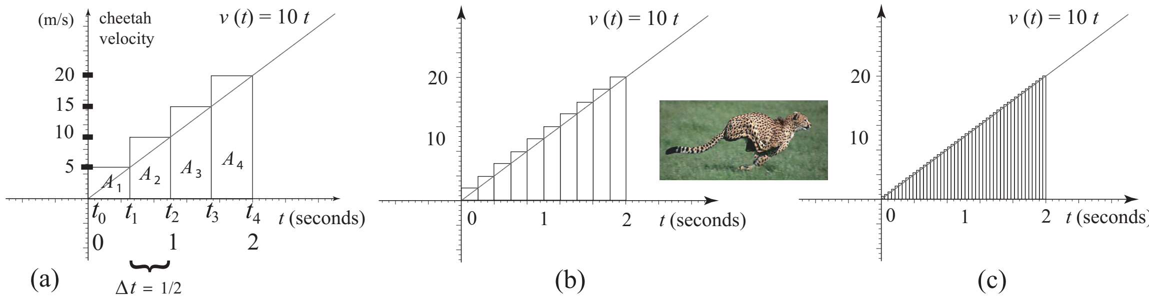

Figure 1.15: To find how far a cheetah runs in 2 seconds, the interval $[0, 2]$ is broken up into subintervals of duration Δt. In (a) the subintervals are $[0, \frac{1}{2}], [\frac{1}{2}, 1], [1, \frac{3}{2}], [\frac{3}{2}, 2]$. The motion over a short subinterval is approximately uniform, so that the distance traveled over Δt can be estimated by the area of a rectangular box. As the number of boxes grows in (b) and (c), the sum of their areas gets closer to the actual area under the graph of $v(t)$ on $[0, 2]$.

1, $t_3 = \frac{3}{2}$, and $t_4 = 2$. How far the cheetah runs during each subinterval is approximated by the area A_k of the corresponding rectangle of height $v(t_k)$ and width Δt,

$$A_k = v(t_k)\Delta t, \tag{1.38}$$

representing uniform motion with constant velocity $v(t_k)$ over that subinterval, with $k = 1, 2, 3, 4$ in (a). Then the total distance run over $[0, 2]$ is approximated by the sum of the areas of the rectangles,

$$A_1 + A_2 + A_3 + A_4 = v(t_1)\Delta t + v(t_2)\Delta t + v(t_3)\Delta t + v(t_4)\Delta t. \tag{1.39}$$

With $v(t_1) = 5, v(t_2) = 10, v(t_3) = 15, v(t_4) = 20$, the estimated distance run is $(5 + 10 + 15 + 20) \cdot \frac{1}{2} = 25$ meters. As the number of boxes grows in Figure 1.15, or as $\Delta t \to 0$, the region covered by the boxes becomes indistinguishable from the triangular region under $v(t)$ on $[0, 2]$. With 10 boxes in (b), the estimated total distance run is $(2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20) \cdot \frac{1}{5} = 22$ meters. In the limit as the number of boxes approaches ∞, it is reasonable (and correct) to conclude that the total distance run by the cheetah is equal to the area under the graph of $v(t)$ on $[0, 2]$. Since this region is a right triangle with base 2 and height 20, the distance run is $\frac{1}{2} \times \textit{base} \times \textit{height} = \frac{1}{2} \cdot 2 \cdot 20 = 20$ meters.

Remark. In (1.39), the distance run is approximated by a sum of areas of n rectangles with heights $v(t_k)$ and widths Δt, for $n = 4$. As $n \to \infty$ or $\Delta t \to 0$,

$$v(t_1)\Delta t + v(t_2)\Delta t + ... + v(t_n)\Delta t \quad \longrightarrow \quad \int_0^2 v(t)dt, \tag{1.40}$$

and the limiting sum of areas of infinitesimally thin "rectangles" of height $v(t)$ and width dt is called a **definite integral**. The elongated "S" in (1.35) and in (1.40) is called an *integral sign*. It reminds us that the definite integral is a limit of a *sum* of areas of rectangles, leading to the notation used in (1.35) and (1.40).

Let's obtain the result of Example 1.4.1 using the idea suggested in (1.36). Given $v(t) = 10t$ m/s, we must find $x(t)$ which is an antiderivative of $v(t)$. As in Example 1.2.6,

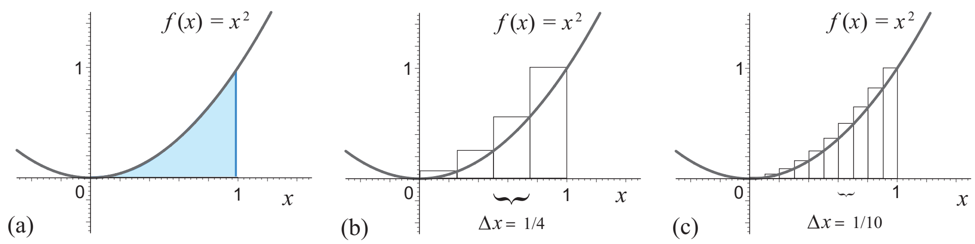

Figure 1.16: The area under the graph of the parabola $f(x) = x^2$ over $[0, 1]$ in (a) can be found from the antiderivative $F(x) = x^3/3 + C$ of $f(x)$. Later, we'll look more closely at how this method is related to summing the areas of the boxes in (b) and (c).

since $\frac{d}{dt}(t^2) = 2t$, we have $\frac{d}{dt}(5t^2) = 10t$, so that $x(t) = 5t^2$ is an antiderivative of $v(t) = 10t$. Thus, via (1.36), the distance run during $[0, 2]$ is the change in position from $t = 0$ to $t = 2$,

$$\int_0^2 v(t)dt = x(2) - x(0) = 5 \cdot 2^2 - 0 = 20 \ \text{ meters.} \tag{1.41}$$

⊕ *The area under the graph of a velocity function $v(t) \geq 0$ on an interval $[a, b]$ is the distance traveled during $[a, b]$, or the difference in the final and initial positions,*

$$\int_a^b v(t)dt = x(b) - x(a) \quad \text{where} \quad \frac{dx}{dt} = v(t). \tag{1.42}$$

Finding the area under the graph of $v(t)$ on $[a, b]$ reduces to finding an antiderivative $x(t)$ of $v(t)$ with respect to t, written as

$$x(t) = \int v(t)dt \qquad \Longleftrightarrow \qquad \frac{dx}{dt} = v(t). \tag{1.43}$$

The expression for $x(t)$ is called an **indefinite integral**, and denotes any *function* which is an antiderivative of $v(t)$. In (1.36) and (1.41) the definite integral involves a specific interval such as $[0, 10]$ or $[0, 2]$, and results in a *number*. The indefinite integral is a function $x(t)$.

Example 1.4.2. Let $v(t) = t^2$ feet/second be the velocity of a particle which starts at the point $x = 2$ at $t = 0$. (a) Find the position $x(t)$. (b) Find how far the particle travels during the time interval $[1, 4]$.

Solution: (a) From (1.16), $\frac{d}{dt}(t^3) = 3t^2$. Then the indefinite integral, or antiderivative, of $v(t)$ is

$$x(t) = \int v(t)dt = \int t^2 dt = \frac{t^3}{3} + C, \tag{1.44}$$

where any constant C can be added to $t^3/3$, and the result is still an antiderivative of t^2, since $\frac{d}{dt}(\frac{t^3}{3} + C) = t^2 + 0 = t^2$. Since the particle starts at $x = 2$, we must have $x(0) = 0 + C = 2$, or $C = 2$. Then $x(t) = \frac{t^3}{3} + 2$. (b) The distance traveled by the particle is the area under $v(t) = t^2$ over the interval $[1, 4]$, or equivalently the change in position from $t = 1$ to $t = 4$,

$$\int_1^4 v(t)dt = \int_1^4 t^2 dt = x(4) - x(1). \tag{1.45}$$

Evaluating the difference in position yields the total distance as $x(4) - x(1) = \left(\frac{64}{3} + 2\right) - \left(\frac{1}{3} + 2\right) = \frac{63}{3} = 21$ feet.

Our analysis of motion is particularly important because the method it yields to find area works in general. Let's find the area $\int_0^1 x^2 dx$ of the shaded region under the parabola $f(x) = x^2$ over $[0, 1]$ in Figure 1.16. This region is not a shape whose area is known from elementary geometry. Any antiderivative $F(x)$ of $f(x) = x^2$ can be written as

$$F(x) = \int x^2 dx = \frac{x^3}{3} + C. \tag{1.46}$$

As suggested by (1.45), we now find the change in $F(x)$ from $x = 0$ to $x = 1$,

$$\int_0^1 x^2 dx = F(1) - F(0) = \left(\frac{1^3}{3} + C\right) - \left(\frac{0^3}{3} + C\right) = \frac{1}{3}\,. \tag{1.47}$$

Let's collect our results for antiderivatives of powers of x. First, from (1.16) and (1.17),

$$\frac{d}{dx}(x) = 1, \quad \frac{d}{dx}\left(\frac{x^2}{2}\right) = x, \quad \frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2, \quad ..., \quad \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, ... \,. \tag{1.48}$$

As indefinite integrals, these results can be written

$$\int 1\, dx = x + C, \quad \int x\, dx = \frac{x^2}{2} + C, \quad \int x^2\, dx = \frac{x^3}{3} + C, \quad ..., \quad \int x^n\, dx = \frac{x^{n+1}}{n+1} + C, ... \,. \tag{1.49}$$

Integration, both definite and indefinite, obeys the properties of differentiation exhibited in Equations (1.18), (1.19) and (1.20).

⊕ *The integral of a constant times a function is the constant times the integral; the integral of a sum or difference is the sum or difference of the integrals.*

Example 1.4.3. (a) Find an indefinite integral, or antiderivative $F(x)$, of $f(x) = 4x^3 + x + 2$. (b) Find the area under the graph of $f(x)$ on the interval $[1, 2]$.

Solution: (a)
$$\begin{aligned} F(x) &= \int (4x^3 + x + 2)\, dx = \int 4x^3 dx + \int x dx + \int 2dx \\ &= 4\int x^3 dx + \int x dx + 2\int 1 dx = x^4 + \frac{x^2}{2} + 2x + C. \end{aligned}$$

(b) $$\int_1^2 (4x^3 + x + 2)\, dx = F(2) - F(1) = (16 + 2 + 4 + C) - \left(1 + \frac{1}{2} + 2 + C\right) = 18\frac{1}{2}$$

The next result will be proven later, but should be reasonable from the above.

Theorem 1.2 (Antiderivative of any polynomial).

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n = 0, 1, 2, 3, ..., \quad C \in \mathbb{R}, \tag{1.50}$$

and for any n^{th} order polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$,

$$\int p(x) dx = \frac{a_n}{n+1} x^{n+1} + \frac{a_{n-1}}{n} x^n + \ ... \ + \frac{a_1}{2} x^2 + a_0 x + C, \quad C \in \mathbb{R}. \tag{1.51}$$

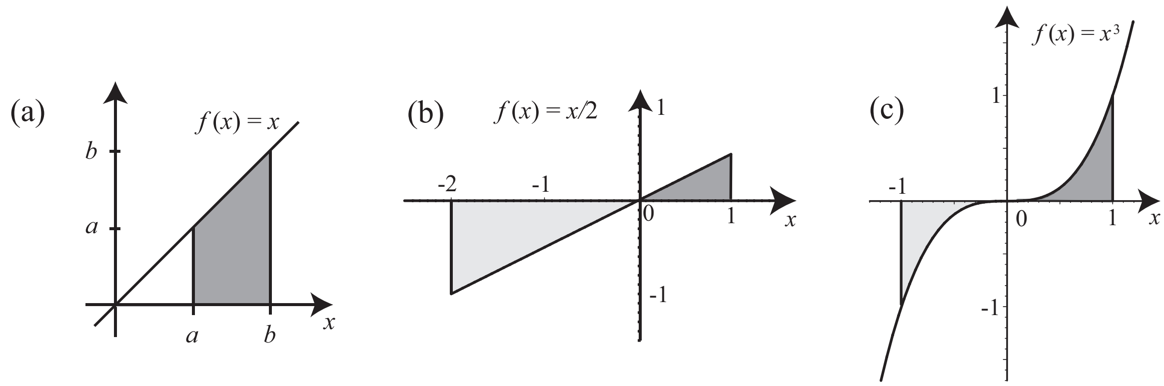

Figure 1.17: In (a), the area of the dark grey region is the area $\frac{b^2}{2}$ of the triangle with base b and height b *minus* the area $\frac{a^2}{2}$ of the triangle with base a and height a. In (b), the total signed area is the sum of the negative light grey area and the positive dark grey area, with a negative net result. In (c), the signed area of $g(x) = x^3$ on $[-1, 1]$ is 0, since the positive area is exactly balanced by the negative area.

Analogous to (1.40), the area under the graph of a function $f(x)$ on $[a, b]$ can be approximated by a sum of n rectangular areas as in Figure 1.16,

$$\int_a^b f(x)dx \;\approx\; f(x_1)\Delta x + f(x_2)\Delta x + ... + f(x_n)\Delta x, \tag{1.52}$$

with the exact result obtained as $\Delta x \to 0$ or $n \to \infty$. We'll examine this procedure in much more detail later, but based on our findings with velocity and the result in (1.47), the following should be reasonable.

Theorem 1.3 (Definite integral of any polynomial). *For any n^{th} order polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$,*

$$\int_a^b f(x)dx = F(b) - F(a), \quad \frac{dF}{dx} = f(x), \quad a, b \in \mathbb{R},\ a \le b. \tag{1.53}$$

Theorem 1.3 is part of the **Fundamental Theorem of Calculus**, which will be proved later. This result, and its generalizations, serve as pillars of calculus. Moreover, this theorem shows that the two main problems of calculus are closely related.

Remark. So far we have restricted our discussions to positive functions. However, where a function takes negative values, the contribution to the definite integral is negative. For example, in Figure 1.17 (b), the definite integral over $[-2, 0]$ is the *negative* of the area of the light grey triangle. The definite integral over $[0, 1]$, where the function is positive, is just the area of the dark grey triangle. The definite integral over the whole interval $[-2, 1]$ is called the **signed area**, which equals the area of the dark grey region minus the area of the light grey region. If the positive area equals the negative area, as in Figure 1.17 (c), the definite integral, or signed area, is 0.

Example 1.4.4. Find (a) $\displaystyle\int_a^b x dx$ (b) $\displaystyle\int_{-2}^1 \frac{1}{2}x dx$ (c) $\displaystyle\int_{-1}^1 x^3 dx$ (d) $\displaystyle\int_0^3 (x^2 + 2x^3)dx$.

Solution: (a)Any antiderivative of x can be written as $x^2/2 + C$. Thus by Theorem 1.3,

$$\int_a^b x dx = \frac{b^2}{2} - \frac{a^2}{2} \,, \tag{1.54}$$

which is verified geometrically in Figure 1.17 (a). (b) By Theorem 1.3,

$$\int_{-2}^{1} \frac{x}{2}\, dx = \frac{1}{2}\left(\frac{1^2}{2} - \frac{(-2)^2}{2}\right) = -\frac{3}{4}\,.$$

(c) Since the positive area exactly balances the negative area, as shown in Figure 1.17 (c), we expect $\int_{-1}^{1} x^3 dx = 0$. Theorem 1.3 yields

$$\int_{-1}^{1} x^3 dx = \frac{1^4}{4} - \frac{(-1)^4}{4} = 0.$$

(d)
$$\int_0^3 (x^2 + 2x^3)\, dx = \int_0^3 x^2\, dx + 2\int_0^3 x^3\, dx = \frac{3^3}{3} \;+\; 2\cdot\frac{3^4}{4} = \frac{99}{2}\,.$$

BRIEF SUMMARY: The signed area enclosed by the graph of a function $f(x)$ on an interval $[a, b]$ is found from the definite integral of f on the interval. By analyzing motion, we found that the area can be obtained from an antiderivative of $f(x)$, which links the two fundamental problems of calculus.

Exercises

1. A train is speeding down a track, as in Figure 1.10, with a constant velocity of $v = 220$ feet/second. Find the total distance traveled by the train over a 5 second interval.

2. Let $v(t) = 2t + 1$. (a) Approximate the area under the graph of $v(t)$ on the interval $[0, 2]$ as in Figure 1.15. Split the interval into n subintervals, each of length $\Delta t = 2/n$, and perform the sum in (1.40) with $t_0 = 0$, $t_1 = \Delta t$, $t_2 = 2\Delta t$, and $t_j = j\Delta t$ in general. Carry out the approximation for the three cases $n = 2, 4, 8$. (b) Carry out the steps of (a) but use the left endpoints of each subinterval to determine the height of each rectangle. That is, perform the sum $v(t_0)\Delta t + v(t_1)\Delta t + ... + v(t_{n-1})\Delta t$ in each case. (c) Calculate the area under the graph by finding the distance traveled, $\int_0^2 v(t)dt = x(2) - x(0)$, and compare your results with (a) and (b). (d) Find the area under the graph of $v(t)$ on $[0, 2]$ from elementary geometry, knowing the area of rectangles and right triangles.

3. Carry out steps (a)-(c) of the previous exercise for each function and interval given. (a) $v(t) = 2 - t$ on $[0, 2]$. (b) $v(t) = t^2$ on $[0, 2]$. (c) $v(t) = -\frac{1}{2}t + 3$ on $[-1, 1]$. (d) $v(t) = t^2 + 2$ on $[-2, 0]$.

4. For each of the following velocity functions $v(t)$ meters/second, find the distance traveled in meters over the given time interval $[a, b]$ in seconds.

   (a) $v(t) = 3t + 1$ on [1,3] (b) $v(t) = 4 + \frac{1}{4}t^2$ on [0,6] (c) $v(t) = 3t^2$ on [0,1]

   (d) $v(t) = -10t + 20$ on [0,2] (e) $v(t) = t^6 + 3t^3 + 3t + 1$ on $[0, 1\frac{1}{4}]$

5. A steam catapult aboard an aircraft carrier can accelerate an F-18 Hornet so that its velocity is given by $v(t) = 104.85t$ feet/second. If the jet reaches its take-off velocity of 173 miles per hour at the end of the runway, how long does it take for the jet to take off, and how long is the runway?

6. Find the general form for any antiderivative of each of the following functions.

   (a) $f(x) = x^2$ (b) $g(x) = 5x^2$ (c) $f(x) = 3x - 4$ (d) $u(x) = x^3 + 2$

   (e) $g(x) = x + \pi$ (f) $f(x) = x^5 - 2x^3 + x$ (g) $v(x) = a^2 - x^2,\ a \in \mathbb{R}$

7. (a) An object's velocity is given by $v(t) = 2 - 3t^2/2$. If its initial position is $x(0) = 0$, find $x(t)$. (b) An object's velocity is given by $v(t) = 5t^3 - 2t^2 + t + 1$. If its initial position is $x(0) = 2$, find $x(t)$.

8. In each case find the antiderivative $F(x) = \int f(x)dx$ which satisfies the given condition.
   (a) $f(x) = x - x^3,\ F(0) = 1$ (b) $f(x) = 1 + 3x^2,\ F(1) = -1$
   (c) $f(x) = \pi - x^{20} + 54x^{12} - x^7,\ F(0) = \pi$ (d) $f(x) = x - x^3/3!,\ F(0) = 1$

9. Find the antiderivative of $(x + 1)(3x - 2)$ that has the value 5 when $x = 0$.

10. In each case calculate a definite integral to find the area under the graph for the given function on the interval $[0, b]$. (a) $\frac{1}{2}x + 1$ (b) $3x^2$ (c) $3x^3 + x$ (d) $x^2 + 5x^4$

11. Find the following definite integrals.

    (a) $\int_a^b (2x - 1)\, dx$ (b) $\int_0^1 (x^2 + 2x + 1)\, dx$ (c) $\int_{-1}^2 \frac{1}{2}x\, dx$ (d) $\int_{-2}^2 \frac{1}{3}x^5\, dx$

    (e) $\int_1^5 (x^4 - 3x^2 + 14)\, dx$ (f) $\int_{-\sqrt{2}}^{\sqrt{2}} (\frac{1}{2}x^2 + 1)\, dx$ (g) $\int_0^1 (x^n - x^{n+1})\, dx,\ n \geq 1$

12. Find the following definite integrals.

    (a) $\int_a^b (3t - c)\, dt$ (b) $\int_0^1 (t^3 + 2t - 3)\, dt$ (c) $\int_{-2}^1 \frac{1}{3}t^2\, dt$ (d) $\int_{-2}^2 \left(\frac{1}{2}r^3 - r^2 + 5r\right)\, dr$

    (e) $\int_0^1 t\, dx$ (f) $\int_a^b x^2\, dt$ (g) $\int_{-\sqrt{2}}^{\sqrt{2}} (\frac{1}{2}u^2 + 3u)\, du$

13. Find the area of the region bouned by the graphs of $y = x$ and $y = x^2$ on the unit interval $[0, 1]$.

14. Use the formula for the area of a trapezoid to find the area of the shaded region in Figure 1.17 (a). Compare your result with (1.54).

15. Find the total signed area enclosed by the graph of $f(x) = x^5$ and the interval $[-100, 100]$. Do the same for $f(x) = x^3$ and $f(x) = x^2$.

16. Find $\int_{-1}^1 \sqrt{1 - x^2}\, dx$ by considering the region whose area it represents.

17. Consider the interval $[1, 2]$. Find the signed area enclosed by the graph of $f(x) = x + 1$ and the $x-$axis on this interval. Do the same for $g(x) = x^2 + x + \sqrt{2}$ and $u(x) = -2x^3 + \frac{1}{3}x^2 - x$ on this interval.

CHAPTER SYNOPSIS: Calculus involves finding the *rate of change*, or **derivative** of a function $f(x)$ at a point $x = x_0$, and finding the *signed area*, or **definite integral** of a function on an interval $[a, b]$. These problems are addressed by analyzing the function on increasingly *small scales*. Computing the signed area of the graph of a function $f(x)$ involves finding the **antiderivative** of the function $f(x)$, providing a deep connection between the two great problems of calculus. Scientific laws and concepts are usually formulated using derivatives and integrals.