Mathematics 1210 PRACTICE EXAM III Fall 2018

ANSWER KEY

1. Calculate the following integrals:

(a) $\displaystyle\int_0^4 \sqrt{x}\,dx$

Solution:

$$\int_0^4 \sqrt{x}\,dx = \int_0^4 x^{\frac{1}{2}}dx = \left.\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_0^4 = \left.\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^4 = \frac{2}{3}(4^{\frac{3}{2}}) - 0 = \boxed{\frac{16}{3}}.$$

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(b) $\displaystyle\int_0^{\pi/2} \sin x\,dx$

Solution:

$$\int_0^{\pi/2} \sin x\,dx = -\cos x|_0^{\pi/2} = -\cos(\frac{\pi}{2}) - (-\cos(0)) = \boxed{1}.$$

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(c) $\displaystyle\int_1^3 \frac{1-3x^3}{x^2}\,dx$

Solution:

$$\int_1^3 \frac{1-3x^3}{x^2}\,dx = \int_1^3 \frac{1}{x^2} - 3x\,dx = \left.(-\frac{1}{x} - \frac{3x^2}{2})\right|_1^3 = (-\frac{1}{3} - \frac{27}{2}) - (-1 - \frac{3}{2}) = \boxed{-\frac{34}{3}}$$

■

(d) $\displaystyle\int_0^{\pi} \sin^2 x\,dx$

Solution: Notice that

$$\int_0^{\pi} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi} \cos^2 x + \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi} 1\,dx = \boxed{\frac{\pi}{2}}.$$

Or subtract $\cos 2x = \cos^2 x - \sin^2 x$ from $1 = \cos^2 x + \sin^2 x$ to get

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

and integrate to get

$$(\frac{1}{2}x - \frac{1}{2}\sin 2x)|_0^{\pi} = \frac{\pi}{2},$$

which may be about as involved as showing

$$\int_0^{\pi} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi} \cos^2 x + \sin^2 x\,dx.$$

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(e) $\displaystyle\int_0^{\pi} x\cos(x^2+\pi)\,dx$

Solution: Use $u = x^2+\pi$ so $du = 2xdx$ to get

$$\int_{u=\pi}^{u=\pi^2+\pi} \frac{\cos u}{2} du = \frac{1}{2}\sin u\Big|_{\pi}^{\pi^2+\pi} = \boxed{\frac{1}{2}(\sin(\pi^2+\pi))}.$$

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(f) $\displaystyle\int_{-3}^{3} x^3\,dx$

Solution: Notice that the integral of any odd function over an interval which is symmetric about the origin is zero. In this problem, we see that x^3 is odd function and the interval $(-3,3)$ is symmetric about the origin. Hence,

$$\int_{-3}^{3} x^3\,dx = \boxed{0}.$$

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2. Find the general solutions to the following differential equations.

a. $\displaystyle\frac{dy}{dx} = \sqrt[3]{\frac{x}{y}}$

Solution: Separating variables,

$$y^{1/3}dy = x^{1/3}dx$$

and integrating

$$\int y^{1/3}dy = \int x^{1/3}dx$$
$$y^{4/3} = (x^{4/3}+C)$$
$$\boxed{y = (x^{4/3}+C)^{3/4}}.$$

■

(b) $\displaystyle\frac{d^2x}{dt^2} = -\omega^2 x$

Solution: Recall from the previous practice exam that if $x(t) = A\sin(\omega t-\phi)$, then

$$x'(t) = A\cos(\omega t-\phi)\omega = \omega A\cos(\omega t-\phi)$$
$$x''(t) = -\omega^2 A\sin(\omega t-\phi) = -\omega^2 x(t)$$

Hence, we can conclude that the solution of this differential equations is

$$\boxed{x(t) = A\sin(\omega t-\phi)}.$$

Alternatively, we can observe that $x_1(t) = \sin\omega t$ and $x_2(t) = \cos\omega t$ are both solutions, and that any linear combination of the form $x(t) = Ax_1(t) + Bx_2(t)$, for real numbers A and B is also a solution, written in its most general form. ■

(c) $\dfrac{d^2x}{dt^2} = -g$

Solution: Integrating once gives

$$\frac{dx}{dt} = -gt + C,$$

and calling $\frac{dx}{dt}|_{t=0} = v_0$. Integrating again,

$$\boxed{x(t) = -\frac{g}{2}t^2 + v_0 t + x_0}$$

where $x_0 = x(0)$. ■

3. Consider the function

$$f(x) = \sqrt{x}$$

(a) Find the average slope of this function on the interval $(1, 4)$

Solution: The average slope of this function on the interval $(1, 4)$ is given by

$$\frac{f(4) - f(1)}{4 - 1} = \frac{\sqrt{4} - \sqrt{1}}{3} = \boxed{\frac{2}{3}}.$$

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(b) By the Mean Value Theorem, we know there exists a c in the open interval $(1, 4)$ such that $f'(c)$ is equal to this mean slope. What is the value of c in the interval which works.

Solution: Notice that

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

Then, we know that

$$f'(c) = \frac{2}{3}$$
$$\frac{1}{2\sqrt{c}} = \frac{2}{3},$$

which gives $\boxed{c = \frac{9}{16}}$. ■

4. A population of rabbits in a forest is found to grow at a rate proportional to the cube root of the population size. The initial population P is 1000 rabbits, and 5 years later there are 1728 of them.

(a) Write the differential equation for the rabbit population $P(t)$ with the two corresponding conditions.

Solution:

$$\frac{dP}{dt} = kP^{\frac{1}{3}},$$

where k is the proportionality constant, $P(0) = 1000$ and $P(5) = 1728$. ■

(b) Solve this differential equation, that is, find the particular solution which incorporates both conditions.

Solution: Using the differential equation from part (a),

$$\int P^{-\frac{1}{3}} dP = \int k dt,$$

so

$$\frac{3}{2} P^{\frac{2}{3}} = kt + C,$$

where C is the integration constant. Using the initial condition $P(0) = 1000$, we see $C = \frac{3}{2}(1000)^{2/3} = 150$. Then, the second condition $P(5) = 1728$ tells us

$$\frac{3}{2}(1728)^{2/3} = 5k + 150.$$

So, we solve for k to find $k = \frac{66}{5}$. Therefore, $\frac{3}{2}P^{2/3} = \frac{66}{5}t + 150$, so the final solution is

$$\boxed{P(t) = \left(\frac{44}{5}t + 100\right)^{3/2}}.$$

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(c) How long does it take for the rabbit population to quadruple (reach 4000) from its initial value of 1000?

Solution: When $P = 4000$, it follows that

$$(4000)^{2/3} = \frac{66}{5}t + 150,$$

so $\boxed{t = 7.73}$ years. ■

5. Calculate $\int_1^2 (3x^2 - 2)\, dx$ from the definition of the integral, that is, using Riemann sums.Hint: Using $x_i = 1 + \frac{i}{n}$ and $\Delta x = \frac{1}{n}$.) Check your result using the Fundamental Theorem of Calculus.

Solution: Recall that

$$\sum_{i=1}^{n} = \frac{n(n+1)}{2}, \sum_{i=1}^{n} = \frac{n(n+1)(2n+1)}{6}.$$

Using $x_i = 1 + \frac{i}{n}$ and $\Delta x = \frac{1}{n}$, we have

$$\begin{aligned}
\int_1^2 (3x^2 - 2)\, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} (3x_i^2 - 2)\Delta x \\
&= \lim_{n\to\infty} 3\sum_{i=1}^{n} (1 + 2\frac{i}{n} + \frac{i^2}{n^2})\Delta x - 2\sum_{i=1}^{n} \Delta x \\
&= \lim_{n\to\infty} 3\sum_{i=1}^{n} (\frac{1}{n} + 2\frac{i}{n^2} + \frac{i^2}{n^3}) - 2\sum_{i=1}^{n} \frac{1}{n} \\
&= \lim_{n\to\infty} 3(\frac{1}{n}\sum_{i=1}^{n} 1 + \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2) - 2\sum_{i=1}^{n} \frac{1}{n} \\
&= \lim_{n\to\infty} \left[3 + \frac{3(2)(n+1)n}{2n^2} + \frac{3n(n+1)(2n+1)}{6n^3} - 2\right] \\
&= 3 + 3 + 1 - 2 = \boxed{5}.
\end{aligned}$$

From the Fundamental Theorem of Calculus, we have

$$\int_1^2 3x^2 - 2\, dx = x^3 - 2x\Big|_1^2 = (8 - 4) - (1 - 2) = \boxed{5}.$$

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6. Newton's second law for the position $x(t)$ of an object in Earth's gravitational field. is $F = m\frac{d^2x}{dt^2}$, where $F = -mg$, m is the object's mass and $g = 32 ft/s^2$ is the acceleration due to the earth gravity..

   (a) Find $x(t)$ that satisfies the initial conditions $x(0) = x_0$ feet and $v(0) = v_0$ ft/s. (Hints: First solve $\frac{dv}{dt} = -g$, where $v(0) = v_0$ and $v = \frac{dx}{dt}$ where $x(0) = x_0$.)
   Solution: Since $\frac{dv}{dt} = -g$, we can integrate

$$\int dv = \int -g dt$$

$$v(t) = -gt + C.$$

   Using the initial condition $v(0) = v_0$, which gives $c = v_0$. Hence, we hace

$$v(t) = -32t + v_0.$$

   Then, using $\frac{dx}{dt} = v = -32t + v_0$ we integrate

$$\int dx = \int -32t + v_0 dt$$

$$x = -32\frac{t^2}{2} + v_0 t + C.$$

   Using initial condition $x(0) = x_0$, we get $c = x_0$. Hence, the $x(t)$ that satisfies the given initial condition is

$$\boxed{x(t) = -16t^2 + v_0 t + x_0}.$$

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(a) An object is thrown down from a height 64 ft with with velocity $v_0 = -10$ ft/s. How long does it take for the object to hit the ground? (Hint: Use your result from part (a)).

Solution: Using results from part (a),

$$x(t) = -16t^2 + v_0 t + x_0$$

An object throw down from height 64 ft with $v_0 = -10$ ft/s, it implies that $x_0 = 64$ ft. Hence, we get

$$x(t) = -16t^2 - 10t + 64.$$

The object hit the when $x(t) = 0$, it implies that we want to find t such that

$$0 = -16t^2 - 10t + 64.$$

which gives $t = -2.34, 1.71$. Hence, the object will hit the ground at $\boxed{t = 1.71}$ s. ■

7. Find the following: (a) $\dfrac{d}{dx}\displaystyle\int_0^{x^2} \tan\theta \, d\theta$ (b) $\displaystyle\lim_{x\to 0} \frac{\int_0^x (1-\cos t)\, dt}{x^3}$

Solution: (a) Use the Fundamental Theorem and the chain rule. You may want to substitute $u = x^2$ then compute

$$\left(\frac{d}{du}\int_0^u \tan\theta d\theta\right)\left(\frac{d}{dx}x^2\right) = (\tan u)(2x) = \boxed{2x\tan x^2}.$$

(b) $\boxed{\dfrac{1}{6}}$, Use L'Hopital's rule and the Fundamental Theorem. ■

8. Calculate the following:

(a) $\sum_{k=1}^{10}(2^k - 2^{k+1})$

Solution:

$$\sum_{k=1}^{10}(2^k - 2^{k+1}) = 2^1 - 2^2 + 2^2 - 2^3 + ... + 2^{10} - 2^{11} = 2^1 - 2^{11} = \boxed{-2046}.$$

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(b) $\sum_{k=1}^{100}(2k^2 + 1)$

Solution:

$$\sum_{k=1}^{100}(2k^2 + 1) = 2\sum_{k=1}^{100} k^2 + \sum_{k=1}^{100} 1 = 2\frac{100(100+1)(200+1)}{6} + 100 = \boxed{676,700}.$$

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