Mathematics 1210 PRACTICE EXAM II Fall 2018

ANSWER KEY

1. Calculate the following:

a. $\dfrac{d^2x}{dt^2}$, $x(t) = A\sin(\omega t - \phi)$

Solution: Using the chain rule, we have

$$x'(t) = A\cos(\omega t - \phi)\omega = \omega A\cos(\omega t - \phi)$$
$$x''(t) = -\omega^2 A\sin(\omega t - \phi) = -\omega^2 x(t)$$

■

b. $\dfrac{df}{dx}$, $f(x) = \left(\dfrac{x-2}{x-\pi}\right)^3$

Solution: Using the chain rule and the quotient rule, we habe

$$f'(x) = 3\left(\frac{x-2}{x-\pi}\right)^2 \left(\frac{(x-\pi)\frac{d}{dx}(x-2) - (x-2)\frac{d}{dx}(x-\pi)}{(x-\pi)^2}\right)$$
$$= 3\left(\frac{x-2}{x-\pi}\right)^2 \frac{(2-\pi)}{(x-\pi)^2}$$

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c. $\dfrac{dy}{dx}$, $\cos xy = y^2 + 2x$

Solution: The implicit derivative will be used in this problem. Implicit differentiation treats y as a function of x even though it is not known explicitly. First differentiate the equation defining the relationship between x and y with respect to x and apply the usual differentiation rules (chain rule and product rule in the first term) to obtain

$$-\sin(xy)(y + xy') = 2yy' + 2.$$

Then, collect terms which include a factor of y' from the chain rule and factor it out.

$$-y\sin(xy) - 2 = (2y + x\sin(xy))y'$$

Finally, divide to solve for y':

$$\frac{dy}{dt} = y' = \frac{-(2 + y\sin xy)}{(2y + x\sin xy)}$$

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d. $\displaystyle\lim_{x\to 0} \frac{\sin x \tan x}{1 - \cos x}$

Solution: Since

$$\lim_{x\to 0} \frac{\sin x \tan x}{1 - \cos x} = \frac{0}{0} \ ,$$

which tells us that we can apply l'Hopital's Rule. Hence, we have

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin x\tan x}{1-\cos x} &= \lim_{x\to 0}\frac{\frac{d}{dx}(\sin x\tan x)}{\frac{d}{dx}(1-\cos x)}\\
&= \lim_{x\to 0}\frac{(\sin x\sec^2 x)+\tan x\cos x}{(\sin x)}\\
&= \lim_{x\to 0}\frac{(\tan x/\cos x)+\tan x\cos x}{(\sin x)} = \frac{0}{0},
\end{aligned}$$

which once again tells us that we can apply l'Hopital's Rule. So, we have

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin x\tan x}{1-\cos x} &= \lim_{x\to 0}\frac{\frac{d}{dx}(\sin x\tan x)}{\frac{d}{dx}(1-\cos x)}\\
&= \lim_{x\to 0}\frac{\frac{d}{dt}[(\tan x/\cos x)+\tan x\cos x]}{\frac{d}{dt}(\sin x)}\\
&= \lim_{x\to 0}\frac{\frac{\cos x\sec^x - \tan x(-\sin x)}{\cos^2 x}+\tan x(-\sin x)+\cos x\sec^2 x]}{\cos x} = 2.
\end{aligned}$$

Hence, we can conclude that

$$\lim_{x\to 0}\frac{\sin x\tan x}{1-\cos x} = 2.$$

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e. $\dfrac{df}{dx}$, $f(x) = \sin\sqrt{\dfrac{\tan x}{1+x^2}}$

Solution: This is of the form $f(g(\frac{u(x)}{v(x)}))$, where $f(x) = \sin x$, $g(x) = x^{1/2}$, $u(x) = \tan x$, and $v(x) = 1 + x^2$. So, using the chain rule (twice) and quotient rule, we will have

$$f'(x) = \cos\sqrt{\frac{\tan x}{1+x^2}}\;\frac{1}{2}\left(\frac{\tan x}{1+x^2}\right)^{-1/2}\;\frac{(1+x^2)\sec^2 x - 2x\tan x}{(1+x^2)^2}$$

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f. $\dfrac{df}{dx}$, $f(x) = x^2\sin^2(x^3)$

Solution: This is of the form $p(x)q(r(s(x)))$ where $p(x) = x^2$, $q(x) = x^2$, $r(x) = \sin x$, $s(x) = x^3$. So, using the product rule and the chain rule (twice) gives

$$f'(x) = 2x\sin^2(x^3) + 6x^4\sin(x^3)\cos(x^3).$$

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g. $\dfrac{dm}{dv}$, $m(v) = \dfrac{m_0}{\sqrt{1-v^2/c^2}}$, m_0 is rest mass, and $c = 3.0\times 10^8$ m/s.

Solution: Notice that m_0 is constant, and we can rewrite $m(v)$ as

$$m(v) = m_0\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}.$$

Now, we have the function in the form f(g(v)), wo we can use the power rule and chain rule, which gives

$$\frac{dm}{dv} = -\frac{1}{2}m_0\left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}} \cdot \left(-\frac{2v}{c^2}\right) = \frac{m_0 v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}}$$

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h. $\lim_{x\to 0} \frac{1-\cos x}{x}$

Solution: Notice that

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \frac{0}{0},$$

which implies that we can use l' Hopital's Rule. Hence, we have

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}(x)} = \lim_{x\to 0} \frac{\sin x}{1} = 0.$$

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i. $\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$

Solution: Notice that this is the definition of derivative, it follows that

$$\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \frac{d}{dx}x^3 = 3x^3$$

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j. $\lim_{x\to 0} \frac{\sin x - x}{x^3}$

Solution: Using LHopital rule 3 times, we have

$$\begin{aligned}\lim_{x\to 0} \frac{\sin x - x}{x^3} &= \lim_{x\to 0} \frac{\cos x - 1}{3x^2} \\ &= \lim_{x\to 0} \frac{-\sin x}{6x} \\ &= \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}.\end{aligned}$$

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2. A circular oil slick spreads so that its radius increases at the rate of 1.5 feet/second. How fast is the area of the enclosed oil increasing at the end of two hours?

Solution: The area and the radius are related statically by

$$A = \pi r^2$$

In this situation they are related dynamically by

$$A(t) = \pi r(t)^2.$$

Taking derivatives with respect to time using implicit derivative, we get

$$A'(t) = 2\pi r(t) r'(t).$$

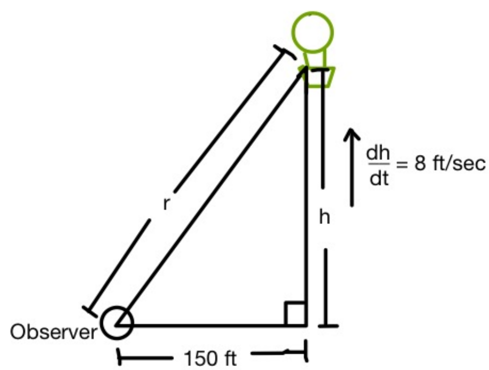
Given $r'(t) = 1.5$ feet/second and $t = 2$ hours $= 7200$ seconds. Hence, after 2 hours the area increasing at rate We get

$$A'(t) = 2\pi(1.5)(1.5)(7200) = 32400\pi.$$

Alternatively, we can directly substitute $r(t) = 1.5t$ into the area formula, $A(t) = 2.25\pi t^2$ and $A'(t) = 4.5\pi t$ so at $t = 2$ hours or 7200 seconds, $A' = 32400$. ■

3. A balloon initially on the ground 150 feet away from an observer is released and rises at a rate of 8 f/s. How fast is the distance between the observer and the balloon changing when the balloon is 50 feet above the ground ?

   Solution: Let r be the distance between observer and balloon and h is the distance of balloon above the ground. Consider the following figure,



The Pythagorean Theorem tells us that

$$r^2 = h^2 + 150^2.$$

Calculating the rate of change of distance between the observer and balloon using implicit derivative and power rule, we get

$$2r\frac{dr}{dt} = 2h\frac{dh}{dt} + 0$$
$$\frac{dr}{dt} = \frac{h}{r}\frac{dh}{dt}.$$

Hence, at $h = 50$, we have $r = \sqrt{50^2 + 150^2} = \sqrt{25000}$. It follows that

$$\frac{dr}{dt} = \frac{50(8)}{\sqrt{25000}} = \frac{400}{\sqrt{25000}}.$$

■

4. Approximate $\sqrt{66}$ and $\sin\frac{\pi}{100}$ using linear approximation (i.e., the differential).

   Solution: Using the approximation of derivative,

$$\frac{dy}{dx} \approx \frac{y(x+\Delta x) - y(x)}{\Delta x}$$

   or

$$y(x+\Delta x) = y(x) + \frac{dy}{dx}\Delta x.$$

   Considering function,

$$y(x) = \sqrt{x}, y'(x) = \frac{1}{2\sqrt{x}}$$

   We have

$$\begin{aligned} y(66) &= y(64+2) \\ &= y(64) + y'(64)(2) \\ &= 8 + \frac{2}{16}. \end{aligned}$$

   So The approximation of $\sqrt{66}$ is $8\frac{2}{16}$.

   In the second one, consider the function

$$y(x) = \sin x, y'(x) = \cos x.$$

   We have

$$\begin{aligned} \sin(\pi/100) &= \sin 0 + \frac{\pi}{100} \\ &= \sin 0 + \cos 0(\frac{\pi}{100}) \\ &= 0 + \frac{\pi}{100} = \frac{\pi}{100}. \end{aligned}$$

   Hence, the approximation of $\sin\frac{\pi}{100}$ is given by $\frac{\pi}{100}$. ■

5. Use the differential to approximate the increase in volume of a spherical bubble as its radius increases from 3 to 3.025 inches.

   Solution: Similar to the previous problems, using the approximation of derivative

$$y(x+\Delta x) = y(x) + y'(x)\Delta x$$

   over function $v = y(r) = \frac{4}{3}\pi r^3$ with $y(r) = 4/3\pi r^3$, $y'(r) = 4\pi r^2$ and $y(3) = 36\pi$, and coincidentally, $y'(3) = 36\pi$. So, we get

$$\begin{aligned} y(0.025) &= y(3.025) - y(3) \\ &= y(3+0.025) - y(3) \\ &= y(3) + y'(3)(0.025) - y(3) \\ &= 36\pi(0.025) \end{aligned}$$

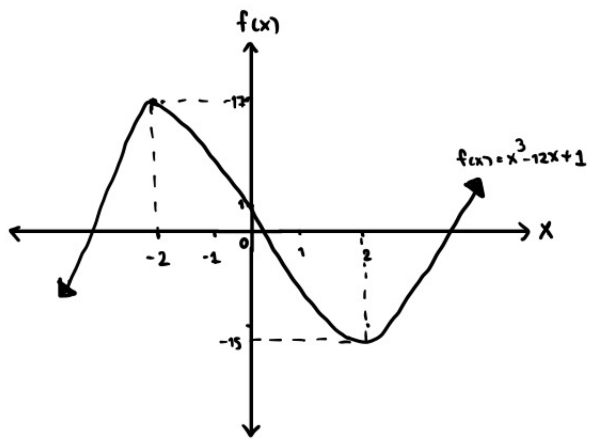
   Hence, the volume of spherical bubble in crease approximately 0.9π cubic inches. ■

6. Consider the following functions. In each cases, find all local maxima and minima of f, where f is increasing and decreasing, where f is concave up and concave down, and all inflection points. Does f have a global maximum or a global minimum? Sketch the graph of $f(x)$.

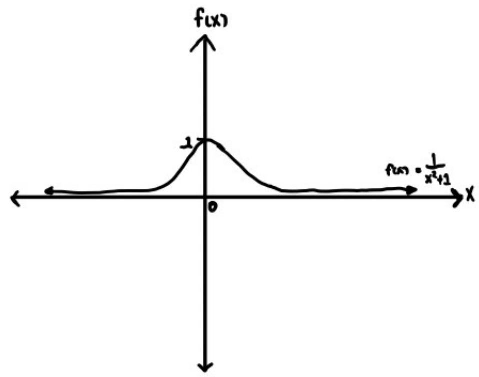
(a) $f(x) = x^3 - 12x + 1$.
Solution: For $f(x) = x^3 - 12x + 1$, $f'(x) = 3x^2 - 12$ and $f''(x) = 6x$. Then $f' = 0$ when $x^2 = 4$ or $x = \pm 2$.
We have f is increasing for $x < -2$, f is decreasing for $-2 < x < 2$ and f is increasing for $x > 2$. Moreover,

$$f(-2) = -8 + 24 + 1 = 17$$
$$f(2) = 8 - 24 + 1 = -15$$

So f has a local maximum at $x = -2$ and a local minimum at $x = 2$. There is no global maximum or minimum. f is concave up where $f'' > 0$, i.e., for $x > 0$, and concave down for $x < 0$. There is one inflection point where f changes concavity, at $x = 0$ since $f''(0) = 0$. From the information, which we obtained we can sketch the graph as following,



(b) $f(x) = \dfrac{1}{1+x^2}$
Solution: For $f(x) = \dfrac{1}{1+x^2}$, $f'(x) = \frac{-2x}{(1+x^2)^2}$. So f is increasing where $f' > 0$, i.e., for $x < 0$ (the denominator of f' is always positive) and f is decreasing where $f' < 0$, i.e., for $x > 0$. There is one local maximum at $x = 0$ where f changes from increasing to decreasing. This is also a global maximum. There are no local or global minima as f approaches zero but remains positive as x tends to infinity in both directions. By the quotient rule, $f''(x) = \frac{6x^2-2}{(1+x^2)^3}$, so $f'' = 0$ and f has inflection points when $x = \pm\frac{\sqrt{3}}{3}$, $f'' < 0$ and f is concave down between these points, and $f'' > 0$ and f is concave up outside these points. From the information, which we obtained we can sketch the graph as following,



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7. Consider $f(x) = 2\sin(x - \frac{\pi}{4})$ on the interval $[\frac{\pi}{2}, \frac{5\pi}{4}]$ Find where f is increasing and decreasing. Find maximum and minimum value of f in the given interval.

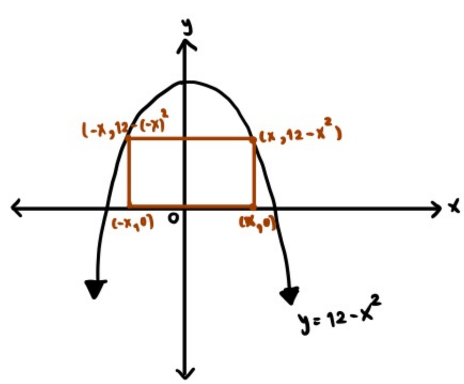
Solution: Notice that

$$f'(x) = 2\cos(x - \frac{\pi}{4}),$$

which is positive for $\frac{\pi}{2} < x < \frac{3\pi}{4}$ (where f is increasing) and negative for $\frac{3\pi}{4} < x < \frac{5\pi}{4}$ (where f is decreasing.) In the interval $[pi, 5pi]$, we know that $f'(x) = 0$ at $x = \frac{3\pi}{4}$. There is a local maximum at $x = \frac{3\pi}{4}$ where $f = 2$, $f' = 0$, and $f'' < 0$, and no other critical points in the interval. Comparing with the endpoints $f(\frac{\pi}{2}) = \sqrt{2}$ and $f(\frac{5\pi}{4}) = 0$, the global maximum is 2 and the global minimum is 0 on this interval. ■

8. A rectangle has two corners on the x-axis and the other two on the parabola $y = 12 - x^2$, with $y \geq 0$. What are the dimensions of the rectangle of this type with maximum area?

Solution: From the problem, we can draew the figure as following,



Since the corners are $(x, 0), (-x, 0)(x, 12 - x^2), (-x, 12 - x^2)$ the area of rectangle will be given by

$$A = 2xy = 2x(12 - x^2) = -2x^3 + 24x.$$

$$\frac{dA}{dx} = 2(12 - 3x^2),$$

which is zero when $x = \pm 2$. We take x to be the vertex on the right, so $x = 2, y = 8$, the dimensions are 4 by 8 (and the maximum area is 32.) ■

9. An object is propelled upward from the ground with initial velocity $v_0 = 32$ f/s. After time t , the height $x(t)$ of the object is given by $x(t) = 16t^2 + v_0 t + x_0$, where x_0 is the initial position, which is assumed to be $x_0 = 0$.

   (a) What is the velocity of the object when it reaches its maximum height?

   Solution: The object will reach its maximum height when the rate of change of height or its velocity is 0 ($v(t) = 0$ f/s . ■

   (b) At what time does the object reach its maximum height?

   Solution: Notice that
   $$v(t) = -32t + v_0.$$
   The object reach its maximum at $v(t) = -32t + v_0 = 0$ or $t = \frac{v_0}{32} = \frac{32}{32} = 1$ sec. ■

   (c) What is the maximum height reached by the object?

   Solution: We know that the object will reach its maximum when $t = 1$ or at height
   $$x(1) = -16 + 32 = 16 \text{ f.}$$
   ■