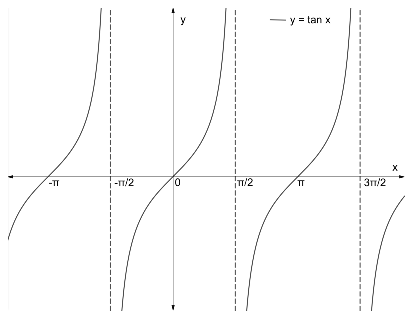
Mathematics 1210 PRACTICE EXAM I Fall 2018

ANSWER KEY

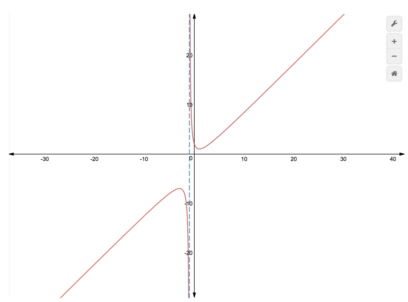
1. Calculate the following limits. If a particular limit does not exist, state this clearly and tell why.

   (a) $\lim_{x\to\sqrt{2}} 3x^2 = 6$

   (b) $\lim_{\theta\to\pi/2} \tan\theta$ does not exist since the limit going to $\pm\infty$ as shown in graph,



   (c) $\lim_{x\to -1} \dfrac{x^2-x+2}{x+1}$ doesn't exist(DNE) because $\frac{x^2-x+2}{x+1}$ goes to $\pm\infty$ at $x=-1$.



   (d) $\lim_{x\to 0^+} \sqrt{x}\sin\left(\dfrac{1}{x^2}\right) = 0$

   (e) $\lim_{x\to 0} \dfrac{\sin(x^2)}{x} = 0.$

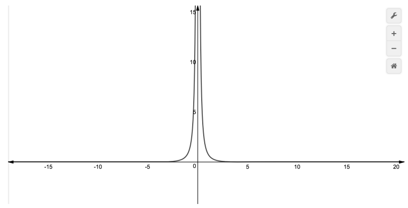
   (f) $\lim_{x\to +\infty} \dfrac{\sin x}{x} = 0$

   (g) $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} x^3, & x\le 2 \\ x, & x>2 \end{cases}$ does not exist (DNE) since LHL is 8 and RHL is 2.

   (h) $\lim_{x\to\pi} f(x)$, where $f(x) = \begin{cases} 0, & x \text{ irrational} \\ \sin\left(\frac{1}{q}\right), & x=\frac{p}{q} \text{ rational} \end{cases} = 0.$

(i) $\lim_{x \to +\infty} \sqrt[3]{\frac{8x^7 + 3x^5}{x^7 + 6x^2}} = 2.$

(j) $\lim_{x \to 0} \frac{\sin x}{x^3}$ does not exist.



2. (a) Let $f(x) = \sqrt{x}$. Using the *definition* of the derivative, calculate $f'(x)$. Do the same for $g(x) = x^2$.

<u>SOLUTION</u>

$$\begin{aligned}
f(x) &= \sqrt{x} \\
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{(x+h)} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{(x+h)} - \sqrt{x}}{h} . \frac{\sqrt{(x+h)} + \sqrt{x}}{\sqrt{(x+h)} + \sqrt{x}} \\
&= \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{(x+h)} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{(x+h)} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$

■

$$\begin{aligned}
g(x) &= x^2 \\
g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h \to 0} 2x + h \\
&= 2x
\end{aligned}$$

■

(b) Using your result from (a), find the equation of the line tangent to the graph of $f(x) = \sqrt{x}$ at $x = 1$. Do the same for $g(x) = x^2$.
<u>Solution</u> The slope for $f(x)$ for $x = 1$ is given by

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2},$$

which is the same as slop of tangent line of $f(x)$ at $x = 1$. The tangent to the graph of $f(x)$ for $x = 1$ is given by

$$\begin{aligned} y - f(1) &= f'(1)(x-1) \\ y - \sqrt{1} &= \frac{1}{2}(x-1) \\ y &= \frac{1}{2}x + \frac{1}{2}. \end{aligned}$$

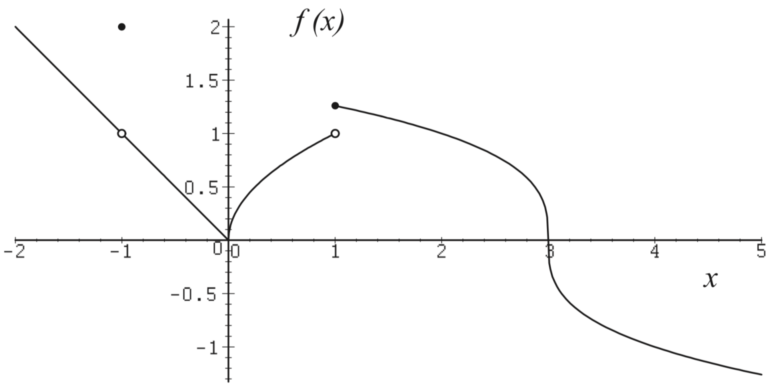
■

Similar, the tangent line of $g(x)$ at $x = 1$ is given by

$$\begin{aligned} y - g(1) &= g'(1)(x-1) \\ y - 1 &= 2x - 2 \\ y &= 2x - 1. \end{aligned}$$

■

3. Let $f(x) = -x$ when $x \le 0, x \ne -1$; $\quad 2$ when $x = -1$; $\quad \sqrt{x}$ when $0 < x < 1$; $\quad \sqrt[3]{3-x}$ when $x \ge 1$. Sketch the graph of $f(x)$.
<u>SOLUTION</u>

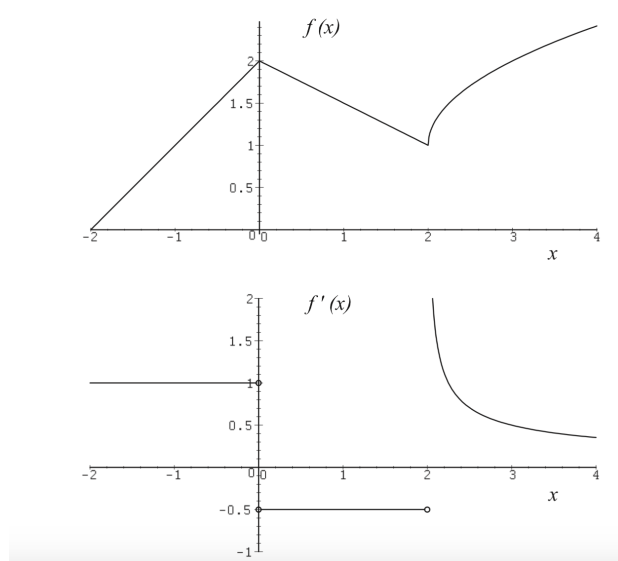


(a) For which points c does $\lim_{x \to c} f(x)$ exist?
<u>SOLUTION</u> : All points except $x = 1$.

(b) For which points is f continuous?
<u>SOLUTION</u> : All points except $x = -1, 1$

(c) For which points is f differentiable?
<u>SOLUTION</u> : all points except $x = -1, 0, 1, 3$

4. Let $f(x) = x + 2$ when $x \le 0$; $\quad -\dfrac{1}{2}x + 2$ when $0 < x \le 2$; $\quad \sqrt{x-2} + 1$ when $x > 2$. Sketch the graph of $f(x)$, and then using your result sketch the graph of $f'(x)$.
<u>SOLUTION</u> :



5. Find the derivative and of

(a) $f(x) = 12x^5 + 5x^4 + x^2 + 2x + 1$

<u>SOLUTION</u> :

$$\begin{aligned} f'(x) &= 12(5)x^{5-1} + 5(4)x^{4-1} + 2x^{2-1} + 2 \\ &= 60x^4 + 20x^3 + 2x + 2. \end{aligned}$$

■

(b) $f(x) = \tan x$

<u>SOLUTION</u> :

$$\begin{aligned} f(x) &= \tan x = \frac{\sin x}{\cos x} \\ f'(x) &= \frac{(\cos x)\frac{d}{dx}\sin x - (\sin x)\frac{d}{dx}\cos x}{(\cos x)^2} \\ &= \frac{\cos x \cos x + \sin x \sin x}{\cos x^2} \\ &= \frac{1}{(\cos x)^2} = \sec^2 x \end{aligned}$$

■

(c) $f(x) = (3x^2 - 2x + 1)(x - 1)$

<u>SOLUTION</u> :

$$\begin{aligned} f'(x) &= (3x^2 - 2x + 1)\frac{d}{dx}(x - 1) + (x - 1)\frac{d}{dx}(3x^2 - 2x + 1) \\ &= (3x^2 - 2x + 1) + (x - 1)(6x - 2) \\ &= 3x^2 - 2x + 1 + 6x^2 - 6x - 2x + 2 \\ &= 9x^2 - 10x + 3 \end{aligned}$$

■

(d) $f(x) = x \cos x$

<u>SOLUTION</u> :

$$\begin{aligned} f'(x) &= x\frac{d}{dx}\cos x + \cos x \frac{d}{dx}x \\ &= -x \sin x + \cos x \end{aligned}$$

■

(e) $f(x) = \dfrac{x^2 + 1}{x + \pi}$

<u>SOLUTION</u> :

$$\begin{aligned} f'(x) &= \frac{(x + \pi)\frac{d}{dx}(x^2 + 1) - (x^2 + 1)\frac{d}{dx}(x + \pi)}{(x + \pi)^2} \\ &= \frac{(x + \pi)(2x) - (x^2 + 1)}{(x + \pi)^2} \\ &= \frac{x^2 + 2\pi x - 1}{x^2 + 2\pi x + \pi^2} \end{aligned}$$

■

6. Let the position $x(t)$ of a particle at time t be given by $x(t) = 3t^2 - 2t + 1$. Find the instantaneous velocity $v(t)$ of the particle for any time t. Where is the particle when its velocity is zero?
<u>Solution</u> : The instantaneous velocity $v(t)$ of the particle for any time t is given by

$$v(t) = x'(t) = 6t - 2$$

Taking $v = 0$ gives

$$0 = 6t - 3,$$

which implies that $t = \frac{1}{3}$ and

$$x(\frac{1}{3}) = 3(\frac{1}{3})^2 - 2(\frac{1}{3}) + 1 = \frac{2}{3}.$$

Hence, $v = 0$ when $t = \dfrac{1}{3}$ and $x = \frac{2}{3}$ ■

7. (a) Find the equation of the line containing the two points $P = (-1, 2)$ and $Q = (1, 1)$ in the form $y = mx + b$.
<u>Solution</u> : The slop of this line is given by

$$m = \frac{2-1}{-1-1} = -\frac{1}{2}.$$

Then, the equation of the line can be calculated as

$$y - 2 = -\frac{1}{2}(x+1)$$
$$y = -\frac{1}{2}x + \frac{3}{2}$$

■

(b) Find the derivative $\dfrac{dy}{dx}$ of the expression for $y(x)$ you found in (a).
<u>Solution</u> : We have

$$\frac{dy}{dx} = -\frac{1}{2}.$$

■