

## Chapter 8

# Vector Products Revisited: A New and Efficient Method of Proving Vector Identities

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### Introduction

The purpose of these remarks is to introduce a *variation on a theme* of the scalar (inner, dot) product and establish multiplication in  $\mathbb{R}^n$ . If  $\mathbf{a} = (a_1, \dots, a_n)$  and  $\mathbf{b} = (b_1, \dots, b_n)$ , we define the product  $\mathbf{ab} \equiv (a_1b_1, \dots, a_nb_n)$ . The product is the multiplication of corresponding vector components as in the scalar product; however, instead of summing the vector components, the product preserves them in vector form. We define the inner sum (or trace) of a vector  $\mathbf{a} = (a_1, \dots, a_n)$  by  $\sigma(\mathbf{a}) = a_1 + \dots + a_n$ . If taken together with an additional definition of cyclic permutations of a vector  ${}^{(p)}\mathbf{a} \equiv (a_{1+p(\bmod n)}, \dots, a_{n+p(\bmod n)})$ , where  $\mathbf{a} \in \mathbb{R}^n$

and the permutation exponent  $p \in \mathbb{Z}$ , we are able to prove complicated vector products (combinations of dot and cross products) extremely efficiently, without appealing to the traditional (and cumbersome) epsilon- $ijk$  proofs. When applied to determinants, this method hints at the rudiments of *Galois* theory.

## Multiplication in $\mathbb{R}^n$

DEFINITION 1 Suppose  $\mathbf{a}$  and  $\mathbf{b} \in \mathbb{R}^n$ , then

$$\mathbf{ab} \equiv (a_1b_1, a_2b_2, \dots, a_nb_n) .$$

The product is the multiplication of corresponding vector components common to the scalar product, however, instead of summing the vector components, the product preserves them in vector form.

THEOREM 1 If  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$  and  $\alpha \in \mathbb{R}$ , then

$$(1.1) \quad \mathbf{a}(\mathbf{bc}) = (\mathbf{ab})\mathbf{c} ,$$

$$(1.2) \quad \mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{ac} ,$$

$$(1.3) \quad \alpha(\mathbf{ab}) = (\alpha\mathbf{a})\mathbf{b} = \mathbf{a}(\alpha\mathbf{b}) ,$$

$$(1.4) \quad \mathbf{1b} = \mathbf{b1} = \mathbf{b} , \text{ where } \mathbf{1} \equiv (1, 1, \dots, 1) \in \mathbb{R}^n ,$$

$$(1.5) \quad \mathbf{ab} = \mathbf{ba} ,$$

$$(1.6) \quad \mathbf{a}\mathbf{\Theta} = \mathbf{\Theta} , \text{ where } \mathbf{\Theta} \equiv (0, 0, \dots, 0) \in \mathbb{R}^n .$$

*Proof: Trivial.*

## Inner Sums and Inner Products

DEFINITION 2 The *inner sum* (or trace) of a vector  $\mathbf{b} \in \mathbb{R}^n$  is defined as

$$\sigma(\mathbf{b}) \equiv \sum_{i=1}^n b_i = b_1 + \dots + b_n .$$

THEOREM 2 If  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$  and  $\alpha \in \mathbb{R}$ , then

$$(2.1) \quad \sigma(\mathbf{a} + \mathbf{b}) = \sigma(\mathbf{a}) + \sigma(\mathbf{b})$$

$$(2.2) \quad \sigma(\mathbf{ab}) = \sigma(\mathbf{ba})$$

$$(2.3) \quad \sigma(\mathbf{c}(\mathbf{a} + \mathbf{b})) = \sigma(\mathbf{ca} + \mathbf{cb})$$

$$(2.4) \quad \sigma(\alpha\mathbf{b}) = \alpha\sigma(\mathbf{b})$$

*Proof: The proofs are straightforward calculations.*

**THEOREM 3** If  $\mathbf{a}$  and  $\mathbf{b} \in \mathbb{R}^n$ , then  $\sigma(\mathbf{ab}) = \mathbf{a} \cdot \mathbf{b}$ , where  $\mathbf{a} \cdot \mathbf{b}$  is the familiar scalar (dot) product.

*Proof:*  $\sigma(\mathbf{ab}) = \sigma(a_1b_1, \dots, a_nb_n) = a_1b_1 + \dots + a_nb_n = \mathbf{a} \cdot \mathbf{b}$ .

**REMARKS** The scalar product can be generalized for  $n$  vectors. In  $\mathbb{R}^3$ , for example,  $\sigma(\mathbf{ab} \cdot \mathbf{c}) = \sigma(\mathbf{ac} \cdot \mathbf{b}) = \sigma(\mathbf{bc} \cdot \mathbf{a})$ . Each of these, expanded by using the inner product, becomes

$$\begin{aligned}\sigma(\mathbf{abc}) &= \mathbf{ab} \cdot \mathbf{c} = |\mathbf{ab}||\mathbf{c}|\cos(\mathbf{ab}, \mathbf{c}) \\ &= \mathbf{ac} \cdot \mathbf{b} = |\mathbf{ac}||\mathbf{b}|\cos(\mathbf{ac}, \mathbf{b}) \\ &= \mathbf{bc} \cdot \mathbf{a} = |\mathbf{bc}||\mathbf{a}|\cos(\mathbf{bc}, \mathbf{a}) ,\end{aligned}$$

respectively. Multiplying these results together,

$$\sigma^3(\mathbf{abc}) = |\mathbf{a}||\mathbf{b}||\mathbf{c}||\mathbf{ab}||\mathbf{ac}||\mathbf{bc}|\cos(\mathbf{a}, \mathbf{bc})\cos(\mathbf{b}, \mathbf{ac})\cos(\mathbf{c}, \mathbf{ab}) .$$

Now, letting  $\mathbf{c} = \mathbf{1}$ , we obtain

$$\sigma^3(\mathbf{ab1}) = \sigma^3(\mathbf{ab}) = \sqrt{n}|\mathbf{a}|^2|\mathbf{b}|^2|\mathbf{ab}|\cos^2(\mathbf{a}, \mathbf{b})\cos(\mathbf{1}, \mathbf{ab}) .$$

Since  $\sigma^2(\mathbf{ab}) = |\mathbf{a}|^2|\mathbf{b}|^2\cos^2(\mathbf{a}, \mathbf{b})$ , we find that an alternative representation of the inner product is given by

$$\sigma(\mathbf{ab}) = \mathbf{a} \cdot \mathbf{b} = \sqrt{n}|\mathbf{ab}|\cos(\mathbf{1}, \mathbf{ab}) .$$

This is more easily seen by the following:  $\mathbf{a} \cdot \mathbf{b} = \mathbf{1} \cdot \mathbf{ab} = |\mathbf{1}||\mathbf{ab}|\cos(\mathbf{1}, \mathbf{ab})$ , which is equivalent to  $\sqrt{n}|\mathbf{ab}|\cos(\mathbf{1}, \mathbf{ab})$ .

A weighted inner product can be defined by  $w_1a_1b_1 + \dots + w_na_nb_n$ , where  $(w_1, \dots, w_n) \in \mathbb{R}^n$  are the weights.

**DEFINITION** If  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{w} \in \mathbb{R}^n$ , where  $\mathbf{w}$  is a weighting vector and the weights  $w_i > 0$ , then the Euclidean weighted inner product of  $\mathbf{a}$  and  $\mathbf{b}$  is defined as

$$\sigma(\mathbf{wab}) .$$

Note that  $\mathbf{w}$  itself may be the product of other vectors, provided that all weights in the final product  $\mathbf{w}$  are positive real numbers.

## PERMUTATION EXPONENTS

In order to represent the cross product in terms of the new product, we define a vector operation that cyclically permutes the vector entries.

DEFINITION 3 If  $\mathbf{b} \in \mathbb{R}^n$  and  $p \in \mathbb{Z}$ , then

$$\langle p \rangle (\mathbf{b}) \equiv (b_{1+p(\text{mod } n)}, b_{2+p(\text{mod } n)}, \dots, b_{n+p(\text{mod } n)}) ,$$

where  $\langle p \rangle$  is the permutation exponent. The cyclic permutation makes the subscript assignment  $i' \rightarrow i + p \pmod{n}$  for each component  $b_i$ . The modulus in the subscript of each component of  $\mathbf{b}$  is there to insure that all subscripts  $i$  satisfy the condition  $1 \leq i \leq n$ .

THEOREM 4 If  $\mathbf{b} \in \mathbb{R}^n$  and  $p, q \in \mathbb{Z}$ , then

$$(4.1) \quad \langle q \rangle (\langle p \rangle \mathbf{b}) = \langle p+q \rangle (\mathbf{b})$$

$$(4.2) \quad \langle q \rangle (\langle p \rangle \mathbf{b}) = \langle p \rangle (\langle q \rangle \mathbf{b})$$

$$(4.3) \quad \langle p \rangle (\mathbf{a} + \mathbf{b}) = \langle p \rangle \mathbf{a} + \langle p \rangle \mathbf{b}$$

$$(4.4) \quad \langle p \rangle (\alpha \mathbf{b}) = \langle p \rangle \alpha \langle p \rangle \mathbf{b}$$

$$(4.5) \quad \langle p \rangle (\alpha \mathbf{b}) = \alpha \langle p \rangle \mathbf{b}$$

*Proof:*

(4.1)  $\langle q \rangle (\langle p \rangle \mathbf{b})$  implies the subscript assignment  $i' \rightarrow i + p \pmod{n}$  followed by the assignment  $i'' \rightarrow i' + q \pmod{n}$ . Since  $i' = i + p \pmod{n}$ , the subscript  $i''$  becomes  $i'' = i + p + q \pmod{n}$ . Since  $p \pmod{n} + q \pmod{n} = p + q \pmod{n}$ , the assignment  $i'' = i + (p + q) \pmod{n}$  is equivalent to  $\langle p+q \rangle \mathbf{b}$ .

(4.2) The process is equivalent to (4.1), except the values  $p$  and  $q$  are interchanged in the assignment  $i'' \rightarrow i + p + q \pmod{n}$ , that is,  $i'' \rightarrow i + q + p \pmod{n}$ , which is equivalent to  $\langle p \rangle (\langle q \rangle \mathbf{b})$ .

(4.3) Note  $\langle p \rangle (\mathbf{a} + \mathbf{b}) = \langle p \rangle (a_1 + b_1, \dots, a_n + b_n)$ , which in turn is equal to

$$(a_{1+p(\text{mod } n)} + b_{1+p(\text{mod } n)}, \dots, a_{n+p(\text{mod } n)} + b_{n+p(\text{mod } n)}) .$$

Now we may write this as

$$(a_{1+p(\text{mod } n)}, \dots, a_{n+p(\text{mod } n)}) + (b_{1+p(\text{mod } n)}, \dots, b_{n+p(\text{mod } n)}) ,$$

which is equivalent to  $\langle p \rangle \mathbf{a} + \langle p \rangle \mathbf{b}$ .

(4.4) Here  $\langle p \rangle (\alpha \mathbf{b}) = \langle p \rangle (\alpha b_1, \dots, \alpha b_n)$  is equivalent to

$$(a_{1+p(\text{mod } n)} b_{1+p(\text{mod } n)}, \dots, a_{n+p(\text{mod } n)} b_{n+p(\text{mod } n)}) .$$

This, in turn, is rewritten as

$$(a_{1+p(\text{mod } n)}, \dots, a_{n+p(\text{mod } n)}) (b_{1+p(\text{mod } n)}, \dots, b_{n+p(\text{mod } n)}) ,$$

which is  $\langle p \rangle \mathbf{a} \langle p \rangle \mathbf{b}$ .

(4.5) In this case,  $\langle p \rangle (\alpha \mathbf{b}) = \langle p \rangle (\alpha b_1, \dots, \alpha b_n)$  which is equivalent to  $\alpha \langle p \rangle \mathbf{b}$  by

$$(\alpha b_{1+p(\text{mod } n)}, \dots, \alpha b_{n+p(\text{mod } n)}) = \alpha (b_{1+p(\text{mod } n)}, \dots, b_{n+p(\text{mod } n)}) .$$

**THEOREM 5** If  $\mathbf{b} \in \mathbb{R}^n$ , then  $\sigma(\mathbf{b}) = \sigma^{(1)}\mathbf{b} = \sigma^{(2)}\mathbf{b} = \dots = \sigma^{(n-1)}\mathbf{b}$ .

*Proof:* Since the order of the components doesn't matter, the sum remains the same for all (cyclic) permutations of the components.

**THEOREM 6** If  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$  and  $p, q, p', q' \in \mathbb{Z}$ , then  $\sigma^{(p)}\mathbf{a} + \sigma^{(q)}\mathbf{b} = \sigma^{(p')}\mathbf{a} + \sigma^{(q')}\mathbf{b}$ .

*Proof:*

$$\begin{aligned}\sigma^{(p)}\mathbf{a} + \sigma^{(q)}\mathbf{b} &= \sigma^{(p)}\mathbf{a} + \sigma^{(q)}\mathbf{b} \\ &= \sigma^{(p')}\mathbf{a} + \sigma^{(q')}\mathbf{b} \\ &= \sigma^{(p')}\mathbf{a} + \sigma^{(q')}\mathbf{b}\end{aligned}$$

**THEOREM 7** If  $\mathbf{a}, \mathbf{b}, \mathbf{1} \in \mathbb{R}^n$ , then  $(\mathbf{ab}) + \sigma^{(1)}(\mathbf{ab}) + \dots + \sigma^{(n-1)}(\mathbf{ab}) = \mathbf{1}\sigma(\mathbf{ab})$ .

*Proof:*  $(\mathbf{ab}) + \sigma^{(1)}(\mathbf{ab}) + \dots + \sigma^{(n-1)}(\mathbf{ab})$

$$\begin{aligned}&= (a_1b_1, \dots, a_nb_n) + (a_2b_2, \dots, a_nb_n, a_1b_1) + \dots + (a_nb_n, a_1b_1, \dots, a_{n-1}b_{n-1}) \\ &= (a_1b_1 + \dots + a_nb_n, a_2b_2 + \dots + a_nb_n + a_1b_1, \dots, a_nb_n + a_1b_1 + \dots + a_{n-1}b_{n-1}) \\ &= (\sigma(\mathbf{ab}), \sigma^{(1)}(\mathbf{ab}), \dots, \sigma^{(n-1)}(\mathbf{ab})) \\ &= (\sigma(\mathbf{ab}), \dots, \sigma(\mathbf{ab})) \\ &= \mathbf{1}\sigma(\mathbf{ab})\end{aligned}$$

## Cross Products

**THEOREM 8** If  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ , then  $\mathbf{a} \times \mathbf{b} = \sigma^{(1)}\mathbf{a}\sigma^{(2)}\mathbf{b} - \sigma^{(2)}\mathbf{a}\sigma^{(1)}\mathbf{b}$ .

*Proof:*

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &\equiv (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \\ &= (a_2b_3, a_3b_1, a_1b_2) - (a_3b_2, a_1b_3, a_2b_1) \\ &= (a_2, a_3, a_1)(b_3, b_1, b_2) - (a_3, a_1, a_2)(b_2, b_3, b_1) \\ &= \sigma^{(1)}\mathbf{a}\sigma^{(2)}\mathbf{b} - \sigma^{(2)}\mathbf{a}\sigma^{(1)}\mathbf{b}.\end{aligned}$$

**THEOREM 9** If  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ , then  $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ .

*Proof:*

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \sigma^{(1)}\mathbf{a}\sigma^{(2)}\mathbf{b} - \sigma^{(2)}\mathbf{a}\sigma^{(1)}\mathbf{b} \\ &= -(\sigma^{(1)}\mathbf{b}\sigma^{(2)}\mathbf{a} - \sigma^{(2)}\mathbf{b}\sigma^{(1)}\mathbf{a}) \\ &= -\mathbf{b} \times \mathbf{a}.\end{aligned}$$

**THEOREM 10** If  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ , then  $\mathbf{a} \times \mathbf{b} = \sigma^{(1)}\mathbf{a}\sigma^{(2)}\mathbf{b} - \sigma^{(2)}\mathbf{a}\sigma^{(1)}\mathbf{b} = \sigma^{(1)}(\mathbf{a}^{(1)}\mathbf{b} - \mathbf{a}^{(2)}\mathbf{b}) = \sigma^{(2)}(\mathbf{a}^{(2)}\mathbf{b} - \mathbf{a}^{(1)}\mathbf{b})$ , by Theorems 4.1, 4.3, and 4.4.

## Vector Identities

The method of proof for the subsequent theorems is as follows: Each vector identity is rewritten in terms of the definitions of the inner product and cross product, by Theorems 3 and 8, respectively. In the case of scalar identities, terms are permuted to isolate any desired vector in its native (un-permuted) form, by Theorem 5. Then the newly formed terms are grouped by similar permutations. It is important to recognize cross product terms,  $\langle 1 \rangle \mathbf{a} \langle 2 \rangle \mathbf{b} - \langle 2 \rangle \mathbf{a} \langle 1 \rangle \mathbf{b}$ , or inner product terms such as  $\langle 1 \rangle (\mathbf{ac}) + \langle 2 \rangle (\mathbf{ac})$ . In the latter case, for example, one adds to this the term  $\mathbf{ac}$  (and subtracts  $\mathbf{ac}$  from another term), for then one recognizes  $\mathbf{ac} + \langle 1 \rangle (\mathbf{ac}) + \langle 2 \rangle (\mathbf{ac})$  as the inner product  $\mathbf{1}(\mathbf{a} \cdot \mathbf{c})$ , according to Theorem 7.

**THEOREM 11** If  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ , then  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ .

*Proof:*

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \sigma(\mathbf{a} \langle 1 \rangle \mathbf{b} \langle 2 \rangle \mathbf{c} - \mathbf{a} \langle 2 \rangle \mathbf{b} \langle 1 \rangle \mathbf{c}) \\ &= \sigma(\langle 2 \rangle \mathbf{ab} \langle 1 \rangle \mathbf{c} - \langle 1 \rangle \mathbf{ab} \langle 2 \rangle \mathbf{c}) \\ &= \sigma(\mathbf{b} \langle 1 \rangle \mathbf{c} \langle 2 \rangle \mathbf{a} - \langle 2 \rangle \mathbf{c} \langle 1 \rangle \mathbf{a}) \\ &= \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \end{aligned}$$

and

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \sigma(\mathbf{a} \langle 1 \rangle \mathbf{b} \langle 2 \rangle \mathbf{c} - \mathbf{a} \langle 2 \rangle \mathbf{b} \langle 1 \rangle \mathbf{c}) \\ &= \sigma(\langle 1 \rangle \mathbf{a} \langle 2 \rangle \mathbf{bc} - \langle 2 \rangle \mathbf{a} \langle 1 \rangle \mathbf{bc}) \\ &= \sigma(\mathbf{c} \langle 1 \rangle \mathbf{a} \langle 2 \rangle \mathbf{b} - \langle 2 \rangle \mathbf{a} \langle 1 \rangle \mathbf{b}) \\ &= \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \end{aligned}$$

**THEOREM 12** If  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ , then  $\mathbf{a} \times \mathbf{b} \times \mathbf{c} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ .

*Proof:*

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \langle 1 \rangle \mathbf{a} \langle 2 \rangle \langle 1 \rangle \mathbf{b} \langle 2 \rangle \mathbf{c} - \langle 2 \rangle \mathbf{b} \langle 1 \rangle \mathbf{c} - \langle 2 \rangle \mathbf{a} \langle 1 \rangle \langle 1 \rangle \mathbf{b} \langle 2 \rangle \mathbf{c} - \langle 2 \rangle \mathbf{b} \langle 1 \rangle \mathbf{c} \\ &= \langle 1 \rangle \mathbf{ab} \langle 1 \rangle \mathbf{c} + \langle 2 \rangle \mathbf{ab} \langle 2 \rangle \mathbf{c} - \langle 1 \rangle \mathbf{a} \langle 1 \rangle \mathbf{bc} - \langle 2 \rangle \mathbf{a} \langle 2 \rangle \mathbf{bc} \\ &= \mathbf{b}(\langle 1 \rangle (\mathbf{ac}) + \langle 2 \rangle (\mathbf{ac})) - \mathbf{c}(\langle 1 \rangle (\mathbf{ab}) + \langle 2 \rangle (\mathbf{ab})) + \mathbf{abc} - \mathbf{abc} \\ &= \mathbf{b}(\mathbf{ac} + \langle 1 \rangle (\mathbf{ac}) + \langle 2 \rangle (\mathbf{ac})) - \mathbf{c}(\mathbf{ab} + \langle 1 \rangle (\mathbf{ab}) + \langle 2 \rangle (\mathbf{ab})) \\ &= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \end{aligned}$$

**THEOREM 13** If  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$ , then  $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$ .

*Proof:*  $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

$$\begin{aligned} &= \sigma(\langle 1 \rangle (\mathbf{ac}) \langle 2 \rangle (\mathbf{bd}) + \langle 2 \rangle (\mathbf{ac}) \langle 1 \rangle (\mathbf{bd}) - \langle 1 \rangle (\mathbf{ad}) \langle 2 \rangle (\mathbf{bc}) - \langle 2 \rangle (\mathbf{ad}) \langle 1 \rangle (\mathbf{bc})) \\ &= \sigma((\mathbf{ac} \langle 1 \rangle (\mathbf{bd}) + \langle 2 \rangle (\mathbf{bd})) - \mathbf{ad}(\langle 1 \rangle (\mathbf{bc}) + \langle 2 \rangle (\mathbf{bc})) + \mathbf{abcd} - \mathbf{abcd}) \end{aligned}$$

$$\begin{aligned}
&= \sigma(\mathbf{ac}(\mathbf{bd} + \langle 1 \rangle(\mathbf{bd}) + \langle 2 \rangle(\mathbf{bd})) - \mathbf{ad}(\mathbf{bc} + \langle 1 \rangle(\mathbf{bc}) + \langle 2 \rangle(\mathbf{bc}))) \\
&= \sigma(\mathbf{ac}(\mathbf{b} \cdot \mathbf{d})) - \sigma(\mathbf{ad}(\mathbf{b} \cdot \mathbf{c})) \\
&= (\mathbf{b} \cdot \mathbf{d})\sigma(\mathbf{ac}) - (\mathbf{b} \cdot \mathbf{c})\sigma(\mathbf{ad}) \\
&= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})
\end{aligned}$$

**THEOREM 14** If  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$ , then  $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{b}(\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d})) - \mathbf{a}(\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d}))$ .

*Proof:* Let  $(\mathbf{c} \times \mathbf{d}) = \mathbf{e}$ , then  $(\mathbf{a} \times \mathbf{b}) \times \mathbf{e}$

$$\begin{aligned}
&= \langle 2 \rangle \mathbf{ab} \langle 2 \rangle \mathbf{e} - \mathbf{a} \langle 2 \rangle \mathbf{b} \langle 2 \rangle \mathbf{e} - \mathbf{a} \langle 1 \rangle \mathbf{b} \langle 1 \rangle \mathbf{e} + \langle 1 \rangle \mathbf{ab} \langle 1 \rangle \mathbf{e} \\
&= \mathbf{b} \langle 1 \rangle (\mathbf{ae}) + \langle 2 \rangle (\mathbf{ae}) - \mathbf{a} \langle 1 \rangle \mathbf{b} \langle 1 \rangle \mathbf{e} + \langle 2 \rangle (\mathbf{be}) + \mathbf{abe} - \mathbf{abe} \\
&= \mathbf{b}(\mathbf{ae} + \langle 1 \rangle(\mathbf{ae}) + \langle 2 \rangle(\mathbf{ae})) - \mathbf{a}(\mathbf{be} + \langle 1 \rangle(\mathbf{be}) + \langle 2 \rangle(\mathbf{be})) \\
&= \mathbf{b}(\mathbf{a} \cdot \mathbf{e}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{e}) \\
&= \mathbf{b}(\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d})) - \mathbf{a}(\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d}))
\end{aligned}$$

**THEOREM 15** If  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$ , then  $\mathbf{a} \times (\mathbf{b} \times (\mathbf{c} \times \mathbf{d})) = \mathbf{b}(\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d})) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \times \mathbf{d})$ .

*Proof:* Let  $(\mathbf{c} \times \mathbf{d}) = \mathbf{e}$ , then  $\mathbf{a} \times (\mathbf{b} \times \mathbf{e})$

$$\begin{aligned}
&= \langle 1 \rangle \mathbf{a} \langle 2 \rangle \langle 1 \rangle \mathbf{b} \langle 2 \rangle \mathbf{e} - \langle 2 \rangle \mathbf{b} \langle 1 \rangle \mathbf{e} - \langle 2 \rangle \mathbf{a} \langle 1 \rangle \langle 1 \rangle \mathbf{b} \langle 2 \rangle \mathbf{e} - \langle 2 \rangle \mathbf{b} \langle 1 \rangle \mathbf{e} \\
&= \langle 1 \rangle \mathbf{ab} \langle 1 \rangle \mathbf{e} - \langle 1 \rangle \mathbf{a} \langle 1 \rangle \mathbf{be} - \langle 2 \rangle \mathbf{a} \langle 2 \rangle \mathbf{be} + \langle 2 \rangle \mathbf{ab} \langle 2 \rangle \mathbf{e} \\
&= \mathbf{b} \langle 1 \rangle (\mathbf{ae}) + \langle 2 \rangle (\mathbf{ae}) - \mathbf{e} \langle 1 \rangle (\mathbf{ab}) + \langle 2 \rangle (\mathbf{ab}) + \mathbf{abe} - \mathbf{abe} \\
&= \mathbf{b}(\mathbf{ae} + \langle 1 \rangle(\mathbf{ae}) + \langle 2 \rangle(\mathbf{ae})) - \mathbf{e}(\mathbf{ab} + \langle 1 \rangle(\mathbf{ab}) + \langle 2 \rangle(\mathbf{ab})) \\
&= \mathbf{b}(\mathbf{a} \cdot \mathbf{e}) - \mathbf{e}(\mathbf{a} \cdot \mathbf{b}) \\
&= \mathbf{b}(\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d})) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \times \mathbf{d})
\end{aligned}$$

**THEOREM 16** If  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ , then  $(\mathbf{a} \times \mathbf{b}) \cdot ((\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})) = (\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}))^2$ .

*Proof:* First,  $(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})$

$$\begin{aligned}
&= \langle 1 \rangle \langle 1 \rangle \mathbf{b} \langle 2 \rangle \mathbf{c} - \langle 2 \rangle \mathbf{b} \langle 1 \rangle \mathbf{c} \langle 2 \rangle \langle 1 \rangle \mathbf{c} \langle 2 \rangle \mathbf{a} - \langle 2 \rangle \mathbf{c} \langle 1 \rangle \mathbf{a} \\
&\quad - \langle 2 \rangle \langle 1 \rangle \mathbf{b} \langle 2 \rangle \mathbf{c} - \langle 2 \rangle \mathbf{b} \langle 1 \rangle \mathbf{c} \langle 1 \rangle \langle 1 \rangle \mathbf{c} \langle 2 \rangle \mathbf{a} - \langle 2 \rangle \mathbf{c} \langle 1 \rangle \mathbf{a} \\
&= (\langle 2 \rangle \mathbf{bc} - \mathbf{b} \langle 2 \rangle \mathbf{c})(\mathbf{c} \langle 1 \rangle \mathbf{a} - \langle 1 \rangle \mathbf{ca}) \\
&\quad - (\mathbf{b} \langle 1 \rangle \mathbf{c} - \langle 1 \rangle \mathbf{bc})(\langle 2 \rangle \mathbf{ca} - \mathbf{c} \langle 2 \rangle \mathbf{a}) \\
&= \langle 1 \rangle \mathbf{a} \langle 2 \rangle \mathbf{bcc} - \mathbf{a} \langle 2 \rangle \mathbf{bc} \langle 1 \rangle \mathbf{c} - \langle 1 \rangle \mathbf{abc} \langle 2 \rangle \mathbf{c} + \mathbf{ab} \langle 1 \rangle \mathbf{c} \langle 2 \rangle \mathbf{c} \\
&\quad - \mathbf{ab} \langle 1 \rangle \mathbf{c} \langle 2 \rangle \mathbf{c} + \langle 2 \rangle \mathbf{abc} \langle 1 \rangle \mathbf{c} + \mathbf{a} \langle 1 \rangle \mathbf{bc} \langle 2 \rangle \mathbf{c} - \langle 2 \rangle \mathbf{a} \langle 1 \rangle \mathbf{bcc} \\
&= \mathbf{cc} \langle 1 \rangle \mathbf{a} \langle 2 \rangle \mathbf{b} - \langle 2 \rangle \mathbf{a} \langle 1 \rangle \mathbf{b} + \mathbf{c} \langle 1 \rangle \mathbf{c} \langle 2 \rangle \mathbf{ab} - \mathbf{a} \langle 2 \rangle \mathbf{b} \\
&\quad + \mathbf{c} \langle 2 \rangle \mathbf{c} \langle 1 \rangle \mathbf{b} - \langle 1 \rangle \mathbf{ab} \\
&= \mathbf{cc} \langle 1 \rangle \mathbf{a} \langle 2 \rangle \mathbf{b} - \langle 2 \rangle \mathbf{a} \langle 1 \rangle \mathbf{b} + \mathbf{c} \langle 1 \rangle (\mathbf{c} \langle 1 \rangle \mathbf{a} \langle 2 \rangle \mathbf{b} - \langle 2 \rangle \mathbf{a} \langle 1 \rangle \mathbf{b})
\end{aligned}$$

$$\begin{aligned}
& +\mathbf{c}^{(2)}(\mathbf{c}^{(1)}\mathbf{a}^{(2)}\mathbf{b} - \langle 2 \rangle \mathbf{a}^{(1)}\mathbf{b}) \\
= & \mathbf{c}(\mathbf{c}(\mathbf{a} \times \mathbf{b})) + \mathbf{c}^{(1)}(\mathbf{c}(\mathbf{a} \times \mathbf{b})) + \mathbf{c}^{(2)}(\mathbf{c}(\mathbf{a} \times \mathbf{b})) \\
= & \mathbf{c}(\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}))
\end{aligned}$$

Then,  $(\mathbf{a} \times \mathbf{b}) \cdot ((\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a}))$

$$\begin{aligned}
& = \sigma((\mathbf{a} \times \mathbf{b})\mathbf{c}(\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}))) \\
& = (\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}))\sigma(\mathbf{c}(\mathbf{a} \times \mathbf{b})) \\
& = (\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}))(\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})) \\
& = (\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}))^2
\end{aligned}$$

## Determinants

DEFINITION 4 The alternating vector  $(1, -1, 1, -1, \dots)$  for the vector space  $\mathbb{R}^n$  is defined as  $\aleph \equiv \sum_{i=1}^n (-1)^{i-1} \hat{e}_i$ , where the  $\hat{e}_i$  are orthonormal vectors.

THEOREM 17 If  $\mathbf{a}, \mathbf{b}, \aleph \in \mathbb{R}^2$ , then  $\det(\mathbf{a}, \mathbf{b}) = \sigma(\aleph \mathbf{a}^{(1)} \mathbf{b})$ .

*Proof:*

$$\begin{aligned}
\det(\mathbf{a}, \mathbf{b}) & = a_1 b_2 - a_2 b_1 \\
& = \sigma((a_1, a_2)(b_2, -b_1)) \\
& = \sigma((1, -1)(a_1, a_2)(b_2, b_1)) \\
& = \sigma(\aleph \mathbf{a}^{(1)} \mathbf{b})
\end{aligned}$$

THEOREM 18 If  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ , then

*Proof:*

$$\begin{aligned}
\det(\mathbf{a}, \mathbf{b}, \mathbf{c}) & = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \\
& = \sigma(\mathbf{a}(\mathbf{b} \times \mathbf{c})) \\
& = \sigma(\mathbf{a}(\langle 1 \rangle \mathbf{b}^{(2)} \mathbf{c} - \langle 2 \rangle \mathbf{b}^{(1)} \mathbf{c}))
\end{aligned}$$

THEOREM 19 If  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \aleph \in \mathbb{R}^4$ , then

$$\begin{aligned}
\det(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) & = \sigma(\aleph \mathbf{a}(\langle 1 \rangle \mathbf{b}^{(2)} \mathbf{c}^{(3)} \mathbf{d} - \langle 3 \rangle \mathbf{c}^{(2)} \mathbf{d} \\
& \quad + \langle 2 \rangle \mathbf{b}^{(3)} \mathbf{c}^{(1)} \mathbf{d} - \langle 1 \rangle \mathbf{c}^{(3)} \mathbf{d} \\
& \quad + \langle 3 \rangle \mathbf{b}^{(1)} \mathbf{c}^{(2)} \mathbf{d} - \langle 2 \rangle \mathbf{c}^{(1)} \mathbf{d}))
\end{aligned}$$

THEOREM 20 If  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e} \in \mathbb{R}^5$ , then

$$\begin{aligned}
 \det(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) = & \sigma(\mathbf{a}^{\langle 1 \rangle} \mathbf{b}^{\langle 2 \rangle} \mathbf{c}^{\langle 3 \rangle} \mathbf{d}^{\langle 4 \rangle} \mathbf{e} - \langle 4 \rangle \mathbf{d}^{\langle 3 \rangle} \mathbf{e}) \\
 & + \langle 3 \rangle \mathbf{c}^{\langle 4 \rangle} \mathbf{d}^{\langle 2 \rangle} \mathbf{e} - \langle 2 \rangle \mathbf{d}^{\langle 4 \rangle} \mathbf{e}) \\
 & + \langle 4 \rangle \mathbf{c}^{\langle 2 \rangle} \mathbf{d}^{\langle 3 \rangle} \mathbf{e} - \langle 3 \rangle \mathbf{d}^{\langle 2 \rangle} \mathbf{e})) \\
 & + \langle 2 \rangle \mathbf{b}^{\langle 3 \rangle} \mathbf{c}^{\langle 1 \rangle} \mathbf{d}^{\langle 4 \rangle} \mathbf{e} - \langle 4 \rangle \mathbf{d}^{\langle 1 \rangle} \mathbf{e}) \\
 & + \langle 4 \rangle \mathbf{c}^{\langle 3 \rangle} \mathbf{d}^{\langle 1 \rangle} \mathbf{e} - \langle 1 \rangle \mathbf{d}^{\langle 3 \rangle} \mathbf{e}) \\
 & + \langle 1 \rangle \mathbf{c}^{\langle 4 \rangle} \mathbf{d}^{\langle 3 \rangle} \mathbf{e} - \langle 3 \rangle \mathbf{d}^{\langle 4 \rangle} \mathbf{e})) \\
 & + \langle 3 \rangle \mathbf{b}^{\langle 4 \rangle} \mathbf{c}^{\langle 1 \rangle} \mathbf{d}^{\langle 2 \rangle} \mathbf{e} - \langle 2 \rangle \mathbf{d}^{\langle 1 \rangle} \mathbf{e}) \\
 & + \langle 1 \rangle \mathbf{c}^{\langle 2 \rangle} \mathbf{d}^{\langle 4 \rangle} \mathbf{e} - \langle 4 \rangle \mathbf{d}^{\langle 2 \rangle} \mathbf{e}) \\
 & + \langle 2 \rangle \mathbf{c}^{\langle 4 \rangle} \mathbf{d}^{\langle 1 \rangle} \mathbf{e} - \langle 1 \rangle \mathbf{d}^{\langle 4 \rangle} \mathbf{e})) \\
 & + \langle 4 \rangle \mathbf{b}^{\langle 1 \rangle} \mathbf{c}^{\langle 3 \rangle} \mathbf{d}^{\langle 2 \rangle} \mathbf{e} - \langle 2 \rangle \mathbf{d}^{\langle 3 \rangle} \mathbf{e}) \\
 & + \langle 2 \rangle \mathbf{c}^{\langle 1 \rangle} \mathbf{d}^{\langle 3 \rangle} \mathbf{e} - \langle 3 \rangle \mathbf{d}^{\langle 1 \rangle} \mathbf{e}) \\
 & + \langle 3 \rangle \mathbf{c}^{\langle 2 \rangle} \mathbf{d}^{\langle 1 \rangle} \mathbf{e} - \langle 1 \rangle \mathbf{d}^{\langle 2 \rangle} \mathbf{e})))
 \end{aligned}$$

According to *Galois* theory, roots of polynomials of degree 5 and higher cannot be characterized by closed-form solutions. Associated with each determinant is a characteristic polynomial which may reveal the connection between *Galois* theory and the determinant representation given by our vector product. A quick comparison of Theorems 19 and 20 reveals that the latter determinant cannot be expressed in terms of only cyclical and reverse-cyclical permutations of the permutation exponents in their natural, ascending order. With further study, we anticipate establishing the connection between the vector product representation and the group-theoretic results of *Galois*.

## Conclusion

The utilization of this method for proving vector identities is reinforced by the very simple rules and notation. When applied to determinants, this method hints at the rudiments of *Galois* theory. Further study in this area by the authors will hopefully establish that connection in the future.