Chapter 8

Vector Products Revisited: A New and Efficient Method of Proving Vector Identities

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Introduction

The purpose of these remarks is to introduce a *variation on a theme* of the scalar (inner, dot) product and establish multiplication in $\mathbb{R}^n$. If $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, we define the product $\mathbf{ab} \equiv (a_1 b_1, \ldots, a_n b_n)$. The product is the multiplication of corresponding vector components as in the scalar product; however, instead of summing the vector components, the product preserves them in vector form. We define the inner sum (or trace) of a vector $\mathbf{a} = (a_1, \ldots, a_n)$ by $\sigma(\mathbf{a}) = a_1 + \cdots + a_n$. If taken together with an additional definition of cyclic permutations of a vector ${}^{\langle p \rangle}\mathbf{a} \equiv (a_{1+p (\bmod n)}, \ldots, a_{n+p (\bmod n)})$, where $\mathbf{a} \in \mathbb{R}^n$

and the permutation exponent $p \in \mathbb{Z}$, we are able to prove complicated vector products (combinations of dot and cross products) extremely efficiently, without appealing to the traditional (and cumbersome) epsilon-*ijk* proofs. When applied to determinants, this method hints at the rudiments of *Galois* theory.

Multiplication in $\mathbb{R}^n$

DEFINITION 1 Suppose $\mathbf{a}$ and $\mathbf{b} \in \mathbb{R}^n$, then

$$\mathbf{ab} \equiv (a_1 b_1, a_2 b_2, \ldots, a_n b_n) \ .$$

The product is the multiplication of corresponding vector components common to the scalar product, however, instead of summing the vector components, the product preserves them in vector form.

THEOREM 1 If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, then

$$\begin{array}{ll}
(1.1) & \mathbf{a}(\mathbf{bc}) = (\mathbf{ab})\mathbf{c} \ , \\
(1.2) & \mathbf{a}(\mathbf{b}+\mathbf{c}) = \mathbf{ab} + \mathbf{ac} \ , \\
(1.3) & \alpha(\mathbf{ab}) = (\alpha\mathbf{a})\mathbf{b} = \mathbf{a}(\alpha\mathbf{b}) \ , \\
(1.4) & \mathbf{1b} = \mathbf{b1} = \mathbf{b} \ , \ \textit{where } \mathbf{1} \equiv (1, 1, \ldots, 1) \in \mathbb{R}^n \ , \\
(1.5) & \mathbf{ab} = \mathbf{ba} \ , \\
(1.6) & \mathbf{a\Theta} = \mathbf{\Theta} \ , \ \textit{where } \mathbf{\Theta} \equiv (0, 0, \ldots, 0) \in \mathbb{R}^n \ .
\end{array}$$

Proof: Trivial.

Inner Sums and Inner Products

DEFINITION 2 The *inner sum* (or trace) of a vector $\mathbf{b} \in \mathbb{R}^n$ is defined as

$$\sigma(\mathbf{b}) \equiv \sum_{i=1}^{n} b_i = b_1 + \ldots + b_n \ .$$

THEOREM 2 If $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, then

$$\begin{array}{ll}
(2.1) & \sigma(\mathbf{a}+\mathbf{b}) = \sigma(\mathbf{a}) + \sigma(\mathbf{b}) \\
(2.2) & \sigma(\mathbf{ab}) = \sigma(\mathbf{ba}) \\
(2.3) & \sigma(\mathbf{c}(\mathbf{a}+\mathbf{b})) = \sigma(\mathbf{ca}+\mathbf{cb}) \\
(2.4) & \sigma(\alpha\mathbf{b}) = \alpha\sigma(\mathbf{b})
\end{array}$$

Proof: The proofs are straightforward calculations.

THEOREM 3 If $\mathbf{a}$ and $\mathbf{b} \in \mathbb{R}^n$, then $\sigma(\mathbf{ab}) = \mathbf{a} \cdot \mathbf{b}$, where $\mathbf{a} \cdot \mathbf{b}$ is the familiar scalar (dot) product.
Proof: $\sigma(\mathbf{ab}) = \sigma(a_1 b_1, \ldots, a_n b_n) = a_1 b_1 + \ldots + a_n b_n = \mathbf{a} \cdot \mathbf{b}$.

REMARKS *The scalar product can be generalized for* n *vectors. In* $\mathbb{R}^3$*, for example,* $\sigma(\mathbf{ab} \cdot \mathbf{c}) = \sigma(\mathbf{ac} \cdot \mathbf{b}) = \sigma(\mathbf{bc} \cdot \mathbf{a})$*. Each of these, expanded by using the inner product, becomes*

$$\begin{aligned} \sigma(\mathbf{abc}) &= \mathbf{ab} \cdot \mathbf{c} = |\mathbf{ab}||\mathbf{c}| cos(\mathbf{ab}, \mathbf{c}) \\ &= \mathbf{ac} \cdot \mathbf{b} = |\mathbf{ac}||\mathbf{b}| cos(\mathbf{ac}, \mathbf{b}) \\ &= \mathbf{bc} \cdot \mathbf{a} = |\mathbf{bc}||\mathbf{a}| cos(\mathbf{bc}, \mathbf{a}) \ , \end{aligned}$$

respectively. Multiplying these results together,

$$\sigma^3(\mathbf{abc}) = |\mathbf{a}||\mathbf{b}||\mathbf{c}||\mathbf{ab}||\mathbf{ac}||\mathbf{bc}| cos(\mathbf{a}, \mathbf{bc}) cos(\mathbf{b}, \mathbf{ac}) cos(\mathbf{c}, \mathbf{ab}) \ .$$

Now, letting $\mathbf{c} = \mathbf{1}$*, we obtain*

$$\sigma^3(\mathbf{ab1}) = \sigma^3(\mathbf{ab}) = \sqrt{n}|\mathbf{a}|^2|\mathbf{b}|^2|\mathbf{ab}| cos^2(\mathbf{a}, \mathbf{b}) cos(\mathbf{1}, \mathbf{ab}) \ .$$

Since $\sigma^2(\mathbf{ab}) = |\mathbf{a}|^2|\mathbf{b}|^2 cos^2(\mathbf{a}, \mathbf{b})$*, we find that an alternative representation of the inner product is given by*

$$\sigma(\mathbf{ab}) = \mathbf{a} \cdot \mathbf{b} = \sqrt{n}|\mathbf{ab}| cos(\mathbf{1}, \mathbf{ab}) \ .$$

This is more easily seen by the following: $\mathbf{a} \cdot \mathbf{b} = \mathbf{1} \cdot \mathbf{ab} = |\mathbf{1}||\mathbf{ab}| cos(\mathbf{1}, \mathbf{ab})$*, which is equivalent to* $\sqrt{n}|\mathbf{ab}| cos(\mathbf{1}, \mathbf{ab})$*.*

A weighted inner product can defined by $w_1 a_1 b_1 + \ldots + w_n a_n b_n$*, where* $(w_1, \ldots, w_n) \in \mathbb{R}^n$ *are the weights.*

DEFINITION If $\mathbf{a}$*,* $\mathbf{b}$*, and* $\mathbf{w} \in \mathbb{R}^n$*, where* $\mathbf{w}$ *is a weighting vector and the weights* $w_i > 0$*, then the Euclidean weighted inner product of* $\mathbf{a}$ *and* $\mathbf{b}$ *is defined as*

$$\sigma(\mathbf{wab}) \ .$$

Note that $\mathbf{w}$ *itself may be the product of other vectors, provided that all weights in the final product* $\mathbf{w}$ *are positive real numbers.*

PERMUTATION EXPONENTS

In order to represent the cross product in terms of the new product, we define a vector operation that cyclically permutes the vector entries.

DEFINITION 3 If $\mathbf{b} \in \mathbb{R}^n$ and $p \in \mathbb{Z}$, then

$$^{\langle p \rangle}(\mathbf{b}) \equiv \left(b_{1+p(mod\ n)}, b_{2+p(mod\ n)}, \ldots, b_{n+p(mod\ n)}\right) ,$$

where $\langle p \rangle$ is the permutation exponent. The cyclic permutation makes the subscript assignment $i' \to i+p\ (mod\, n)$ for each component b_i. The modulus in the subscript of each component of $\mathbf{b}$ is there to insure that all subscripts i satisfy the condition $1 \leq i \leq n$.

THEOREM 4 If $\mathbf{b} \in \mathbb{R}^n$ and $p, q \in \mathbb{Z}$, then

$$\begin{array}{ll} (4.1) & ^{\langle q \rangle}(^{\langle p \rangle}\mathbf{b}) = \ ^{\langle p+q \rangle}(\mathbf{b}) \\ (4.2) & ^{\langle q \rangle}(^{\langle p \rangle}\mathbf{b}) = \ ^{\langle p \rangle}(^{\langle q \rangle}\mathbf{b}) \\ (4.3) & ^{\langle p \rangle}(\mathbf{a}+\mathbf{b}) = \ ^{\langle p \rangle}\mathbf{a} + \ ^{\langle p \rangle}\mathbf{b} \\ (4.4) & ^{\langle p \rangle}(\mathbf{ab}) = \ ^{\langle p \rangle}\mathbf{a}\ ^{\langle p \rangle}\mathbf{b} \\ (4.5) & ^{\langle p \rangle}(\alpha\mathbf{b}) = \ \alpha^{\langle p \rangle}\mathbf{b} \end{array}$$

Proof:

(4.1) $^{\langle q \rangle}(^{\langle p \rangle}\mathbf{b})$ implies the subscript assignment $i' \to i + p(mod\, n)$ followed by the assignment $i'' \to i' + q(mod\, n)$. Since $i' = i + p(mod\, n)$, the subscript i'' becomes $i'' = i + p + q(mod\, n)$. Since $p(mod\, n) + q(mod\, n) = p + q(mod\, n)$, the assignment $i'' = i + (p+q)(mod\, n)$ is equivalent to $^{\langle p+q \rangle}\mathbf{b}$.

(4.2) The process is equivalent to (4.1), except the values p and q are interchanged in the assignment $i'' \to i+p+q(mod\, n)$, that is, $i'' \to i+q+p(mod\, n)$, which is equivalent to $^{\langle p \rangle}(^{\langle q \rangle}\mathbf{b})$.

(4.3) Note $^{\langle p \rangle}(\mathbf{a}+\mathbf{b}) = \ ^{\langle p \rangle}(a_1+b_1, \ldots, a_n+b_n)$, which in turn is equal to

$$\left(a_{1+p(mod\ n)} + b_{1+p(mod\ n)}, \ldots, a_{n+p(mod\ n)} + b_{n+p(mod\ n)}\right) .$$

Now we may write this as

$$\left(a_{1+p(mod\ n)}, \ldots, a_{n+p(mod\ n)}\right) + \left(b_{1+p(mod\ n)}, \ldots, b_{n+p(mod\ n)}\right) ,$$

which is equivalent to $^{\langle p \rangle}\mathbf{a} + \ ^{\langle p \rangle}\mathbf{b}$.

(4.4) Here $^{\langle p \rangle}(\mathbf{ab}) = \ ^{\langle p \rangle}(a_1b_1, \ldots, a_nb_n)$ is equivalent to

$$\left(a_{1+p(mod\ n)}b_{1+p(mod\ n)}, \ldots, a_{n+p(mod\ n)}b_{n+p(mod\ n)}\right) .$$

This, in turn, is rewritten as

$$\left(a_{1+p(mod\ n)}, \ldots, a_{n+p(mod\ n)}\right)\left(b_{1+p(mod\ n)}, \ldots, b_{n+p(mod\ n)}\right) ,$$

which is $^{\langle p \rangle}\mathbf{a}\ ^{\langle p \rangle}\mathbf{b}$.

(4.5) In this case, $^{\langle p \rangle}(\alpha\mathbf{b}) = \ ^{\langle p \rangle}(\alpha b_1, \ldots, \alpha b_n)$ which is equivalent to $\alpha^{\langle p \rangle}\mathbf{b}$ by

$$\left(\alpha b_{1+p(\bmod n)}, \ldots, \alpha b_{n+p(\bmod n)}\right) = \alpha\left(b_{1+p(\bmod n)}, \ldots, b_{n+p(\bmod n)}\right) .$$

THEOREM 5 If $\mathbf{b} \in \mathbb{R}^n$, then $\sigma(\mathbf{b}) = \sigma({}^{\langle 1\rangle}\mathbf{b}) = \sigma({}^{\langle 2\rangle}\mathbf{b}) = \ldots = \sigma({}^{\langle n-1\rangle}\mathbf{b})$.
Proof: Since the order of the components doesn't matter, the sum remains the same for all (cyclic) permutations of the components.

THEOREM 6 If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and $p, q, p', q' \in \mathbb{Z}$, then $\sigma({}^{\langle p\rangle}\mathbf{a} + {}^{\langle q\rangle}\mathbf{b}) = \sigma({}^{\langle p'\rangle}\mathbf{a} + {}^{\langle q'\rangle}\mathbf{b})$.
Proof:

$$\begin{aligned}
\sigma({}^{\langle p\rangle}\mathbf{a} + {}^{\langle q\rangle}\mathbf{b}) &= \sigma({}^{\langle p\rangle}\mathbf{a}) + \sigma({}^{\langle q\rangle}\mathbf{b}) \\
&= \sigma({}^{\langle p'\rangle}\mathbf{a}) + \sigma({}^{\langle q'\rangle}\mathbf{b}) \\
&= \sigma({}^{\langle p'\rangle}\mathbf{a} + {}^{\langle q'\rangle}\mathbf{b})
\end{aligned}$$

THEOREM 7 If $\mathbf{a}, \mathbf{b}, \mathbf{1} \in \mathbb{R}^n$, then $(\mathbf{ab}) + {}^{\langle 1\rangle}(\mathbf{ab}) + \ldots + {}^{\langle n-1\rangle}(\mathbf{ab}) = \mathbf{1}\sigma(\mathbf{ab})$.
Proof: $(\mathbf{ab}) + {}^{\langle 1\rangle}(\mathbf{ab}) + \ldots + {}^{\langle n-1\rangle}(\mathbf{ab})$

$$\begin{aligned}
&= (a_1b_1, \ldots, a_nb_n) + (a_2b_2, \ldots, a_nb_n, a_1b_1) + \ldots + (a_nb_n, a_1b_1, \ldots, a_{n-1}b_{n-1}) \\
&= (a_1b_1 + \ldots + a_nb_n, a_2b_2 + \ldots + a_nb_n + a_1b_1, \ldots, a_nb_n + a_1b_1 + \ldots + a_{n-1}b_{n-1}) \\
&= (\sigma(\mathbf{ab}), \sigma({}^{\langle 1\rangle}(\mathbf{ab})), \ldots, \sigma({}^{\langle n-1\rangle}(\mathbf{ab}))) \\
&= (\sigma(\mathbf{ab}), \ldots, \sigma(\mathbf{ab})) \\
&= \mathbf{1}\sigma(\mathbf{ab})
\end{aligned}$$

Cross Products

THEOREM 8 If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, then $\mathbf{a} \times \mathbf{b} = {}^{\langle 1\rangle}\mathbf{a}^{\langle 2\rangle}\mathbf{b} - {}^{\langle 2\rangle}\mathbf{a}^{\langle 1\rangle}\mathbf{b}$.
Proof:

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &\equiv (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \\
&= (a_2b_3, a_3b_1, a_1b_2) - (a_3b_2, a_1b_3, a_2b_1) \\
&= (a_2, a_3, a_1)(b_3, b_1, b_2) - (a_3, a_1, a_2)(b_2, b_3, b_1) \\
&= {}^{\langle 1\rangle}\mathbf{a}^{\langle 2\rangle}\mathbf{b} - {}^{\langle 2\rangle}\mathbf{a}^{\langle 1\rangle}\mathbf{b}\ .
\end{aligned}$$

THEOREM 9 If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, then $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
Proof:

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= {}^{\langle 1\rangle}\mathbf{a}^{\langle 2\rangle}\mathbf{b} - {}^{\langle 2\rangle}\mathbf{a}^{\langle 1\rangle}\mathbf{b} \\
&= -({}^{\langle 1\rangle}\mathbf{b}^{\langle 2\rangle}\mathbf{a} - {}^{\langle 2\rangle}\mathbf{b}^{\langle 1\rangle}\mathbf{a}) \\
&= -\mathbf{b} \times \mathbf{a}\ .
\end{aligned}$$

THEOREM 10 If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, then $\mathbf{a} \times \mathbf{b} = {}^{\langle 1\rangle}\mathbf{a}^{\langle 2\rangle}\mathbf{b} - {}^{\langle 2\rangle}\mathbf{a}^{\langle 1\rangle}\mathbf{b} = {}^{\langle 1\rangle}(\mathbf{a}^{\langle 1\rangle}\mathbf{b} - {}^{\langle 1\rangle}\mathbf{ab}) = {}^{\langle 2\rangle}({}^{\langle 2\rangle}\mathbf{ab} - \mathbf{a}^{\langle 2\rangle}\mathbf{b})$, by Theorems 4.1, 4.3, and 4.4.

Vector Identities

The method of proof for the subsequent theorems is as follows: Each vector identity is rewritten in terms of the definitions of the inner product and cross product, by Theorems 3 and 8, respectively. In the case of scalar identities, terms are permutated to isolate any desired vector in its native (un-permutated) form, by Theorem 5. Then the newly formed terms are grouped by similar permutations. It is important to recognize cross product terms, ${}^{\langle 1\rangle}\mathbf{a}^{\langle 2\rangle}\mathbf{b} - {}^{\langle 2\rangle}\mathbf{a}^{\langle 1\rangle}\mathbf{b}$, or inner product terms such as ${}^{\langle 1\rangle}(\mathbf{ac}) + {}^{\langle 2\rangle}(\mathbf{ac})$. In the latter case, for example, one adds to this the term $\mathbf{ac}$ (and subtracts $\mathbf{ac}$ from another term), for then one recognizes $\mathbf{ac} + {}^{\langle 1\rangle}(\mathbf{ac}) + {}^{\langle 2\rangle}(\mathbf{ac})$ as the inner product $\mathbf{1}(\mathbf{a}\cdot\mathbf{c})$, according to Theorem 7.

THEOREM 11 If $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{R}^3$, then $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$.
Proof:

$$\begin{aligned}
\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= \sigma(\mathbf{a}^{\langle 1\rangle}\mathbf{b}^{\langle 2\rangle}\mathbf{c} - \mathbf{a}^{\langle 2\rangle}\mathbf{b}^{\langle 1\rangle}\mathbf{c}) \\
&= \sigma({}^{\langle 2\rangle}\mathbf{ab}^{\langle 1\rangle}\mathbf{c} - {}^{\langle 1\rangle}\mathbf{ab}^{\langle 2\rangle}\mathbf{c}) \\
&= \sigma(\mathbf{b}({}^{\langle 1\rangle}\mathbf{c}^{\langle 2\rangle}\mathbf{a} - {}^{\langle 2\rangle}\mathbf{c}^{\langle 1\rangle}\mathbf{a})) \\
&= \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= \sigma(\mathbf{a}^{\langle 1\rangle}\mathbf{b}^{\langle 2\rangle}\mathbf{c} - \mathbf{a}^{\langle 2\rangle}\mathbf{b}^{\langle 1\rangle}\mathbf{c}) \\
&= \sigma({}^{\langle 1\rangle}\mathbf{a}^{\langle 2\rangle}\mathbf{bc} - {}^{\langle 2\rangle}\mathbf{a}^{\langle 1\rangle}\mathbf{bc}) \\
&= \sigma(\mathbf{c}({}^{\langle 1\rangle}\mathbf{a}^{\langle 2\rangle}\mathbf{b} - {}^{\langle 2\rangle}\mathbf{a}^{\langle 1\rangle}\mathbf{b})) \\
&= \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})
\end{aligned}$$

THEOREM 12 If $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{R}^3$, then $\mathbf{a}\times\mathbf{b}\times\mathbf{c} = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$.
Proof:

$$\begin{aligned}
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= {}^{\langle 1\rangle}\mathbf{a}^{\langle 2\rangle}({}^{\langle 1\rangle}\mathbf{b}^{\langle 2\rangle}\mathbf{c} - {}^{\langle 2\rangle}\mathbf{b}^{\langle 1\rangle}\mathbf{c}) - {}^{\langle 2\rangle}\mathbf{a}^{\langle 1\rangle}({}^{\langle 1\rangle}\mathbf{b}^{\langle 2\rangle}\mathbf{c} - {}^{\langle 2\rangle}\mathbf{b}^{\langle 1\rangle}\mathbf{c}) \\
&= {}^{\langle 1\rangle}\mathbf{ab}^{\langle 1\rangle}\mathbf{c} + {}^{\langle 2\rangle}\mathbf{ab}^{\langle 2\rangle}\mathbf{c} - {}^{\langle 1\rangle}\mathbf{a}^{\langle 1\rangle}\mathbf{bc} - {}^{\langle 2\rangle}\mathbf{a}^{\langle 2\rangle}\mathbf{bc} \\
&= \mathbf{b}({}^{\langle 1\rangle}(\mathbf{ac}) + {}^{\langle 2\rangle}(\mathbf{ac})) - \mathbf{c}({}^{\langle 1\rangle}(\mathbf{ab}) + {}^{\langle 2\rangle}(\mathbf{ab})) + \mathbf{abc} - \mathbf{abc} \\
&= \mathbf{b}(\mathbf{ac} + {}^{\langle 1\rangle}(\mathbf{ac}) + {}^{\langle 2\rangle}(\mathbf{ac})) - \mathbf{c}(\mathbf{ab} + {}^{\langle 1\rangle}(\mathbf{ab}) + {}^{\langle 2\rangle}(\mathbf{ab})) \\
&= \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})
\end{aligned}$$

THEOREM 13 If $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \in \mathbb{R}^3$, then $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$.
Proof: $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$

$$\begin{aligned}
&= \sigma({}^{\langle 1\rangle}(\mathbf{ac})^{\langle 2\rangle}(\mathbf{bd}) + {}^{\langle 2\rangle}(\mathbf{ac})^{\langle 1\rangle}(\mathbf{bd}) - {}^{\langle 1\rangle}(\mathbf{ad})^{\langle 2\rangle}(\mathbf{bc}) - {}^{\langle 2\rangle}(\mathbf{ad})^{\langle 1\rangle}(\mathbf{bc})) \\
&= \sigma(\,(\mathbf{ac}({}^{\langle 1\rangle}(\mathbf{bd}) + {}^{\langle 2\rangle}(\mathbf{bd})) - \mathbf{ad}({}^{\langle 1\rangle}(\mathbf{bc}) + {}^{\langle 2\rangle}(\mathbf{bc})) + \mathbf{abcd} - \mathbf{abcd})
\end{aligned}$$

$$\begin{aligned}
&= \sigma(\mathbf{ac}(\mathbf{bd} + {}^{\langle 1\rangle}(\mathbf{bd}) + {}^{\langle 2\rangle}(\mathbf{bd})) - \mathbf{ad}(\mathbf{bc} + {}^{\langle 1\rangle}(\mathbf{bc}) + {}^{\langle 2\rangle}(\mathbf{bc})))\\
&= \sigma(\mathbf{ac}(\mathbf{b}\cdot\mathbf{d})) - \sigma(\mathbf{ad}(\mathbf{b}\cdot\mathbf{c}))\\
&= (\mathbf{b}\cdot\mathbf{d})\sigma(\mathbf{ac}) - (\mathbf{b}\cdot\mathbf{c})\sigma(\mathbf{ad})\\
&= (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})
\end{aligned}$$

THEOREM 14 If $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \in \mathbb{R}^3$, then $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \mathbf{b}(\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})) - \mathbf{a}(\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d}))$.

Proof: Let $(\mathbf{c}\times\mathbf{d}) = \mathbf{e}$, then $(\mathbf{a}\times\mathbf{b})\times\mathbf{e}$

$$\begin{aligned}
&= {}^{\langle 2\rangle}\mathbf{ab}^{\langle 2\rangle}\mathbf{e} - \mathbf{a}^{\langle 2\rangle}\mathbf{b}^{\langle 2\rangle}\mathbf{e} - \mathbf{a}^{\langle 1\rangle}\mathbf{b}^{\langle 1\rangle}\mathbf{e} + {}^{\langle 1\rangle}\mathbf{ab}^{\langle 1\rangle}\mathbf{e}\\
&= \mathbf{b}({}^{\langle 1\rangle}(\mathbf{ae}) + {}^{\langle 2\rangle}(\mathbf{ae})) - \mathbf{a}({}^{\langle 1\rangle}\mathbf{b}^{\langle 1\rangle}\mathbf{e} + {}^{\langle 2\rangle}(\mathbf{be})) + \mathbf{abe} - \mathbf{abe}\\
&= \mathbf{b}(\mathbf{ae} + {}^{\langle 1\rangle}(\mathbf{ae}) + {}^{\langle 2\rangle}(\mathbf{ae})) - \mathbf{a}(\mathbf{be} + {}^{\langle 1\rangle}(\mathbf{be}) + {}^{\langle 2\rangle}(\mathbf{be}))\\
&= \mathbf{b}(\mathbf{a}\cdot\mathbf{e}) - \mathbf{a}(\mathbf{b}\cdot\mathbf{e})\\
&= \mathbf{b}(\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})) - \mathbf{a}(\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d}))
\end{aligned}$$

THEOREM 15 If $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \in \mathbb{R}^3$, then $\mathbf{a}\times(\mathbf{b}\times(\mathbf{c}\times\mathbf{d})) = \mathbf{b}(\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})) - (\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\times\mathbf{d})$.

Proof: Let $(\mathbf{c}\times\mathbf{d}) = \mathbf{e}$, then $\mathbf{a}\times(\mathbf{b}\times\mathbf{e})$

$$\begin{aligned}
&= {}^{\langle 1\rangle}\mathbf{a}^{\langle 2\rangle}({}^{\langle 1\rangle}\mathbf{b}^{\langle 2\rangle}\mathbf{e} - {}^{\langle 2\rangle}\mathbf{b}^{\langle 1\rangle}\mathbf{e}) - {}^{\langle 2\rangle}\mathbf{a}^{\langle 1\rangle}({}^{\langle 1\rangle}\mathbf{b}^{\langle 2\rangle}\mathbf{e} - {}^{\langle 2\rangle}\mathbf{b}^{\langle 1\rangle}\mathbf{e})\\
&= {}^{\langle 1\rangle}\mathbf{ab}^{\langle 1\rangle}\mathbf{e} - {}^{\langle 1\rangle}\mathbf{a}^{\langle 1\rangle}\mathbf{be} - {}^{\langle 2\rangle}\mathbf{a}^{\langle 2\rangle}\mathbf{be} + {}^{\langle 2\rangle}\mathbf{ab}^{\langle 2\rangle}\mathbf{e}\\
&= \mathbf{b}({}^{\langle 1\rangle}(\mathbf{ae}) + {}^{\langle 2\rangle}(\mathbf{ae})) - \mathbf{e}({}^{\langle 1\rangle}(\mathbf{ab}) + {}^{\langle 2\rangle}(\mathbf{ab})) + \mathbf{abe} - \mathbf{abe}\\
&= \mathbf{b}(\mathbf{ae} + {}^{\langle 1\rangle}(\mathbf{ae}) + {}^{\langle 2\rangle}(\mathbf{ae})) - \mathbf{e}(\mathbf{ab} + {}^{\langle 1\rangle}(\mathbf{ab}) + {}^{\langle 2\rangle}(\mathbf{ab}))\\
&= \mathbf{b}(\mathbf{a}\cdot\mathbf{e}) - \mathbf{e}(\mathbf{a}\cdot\mathbf{b})\\
&= \mathbf{b}(\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})) - (\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\times\mathbf{d})
\end{aligned}$$

THEOREM 16 If $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{R}^3$, then $(\mathbf{a}\times\mathbf{b})\cdot((\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})) = (\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}))^2$.

Proof: First, $(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})$

$$\begin{aligned}
&= {}^{\langle 1\rangle}({}^{\langle 1\rangle}\mathbf{b}^{\langle 2\rangle}\mathbf{c} - {}^{\langle 2\rangle}\mathbf{b}^{\langle 1\rangle}\mathbf{c})^{\langle 2\rangle}({}^{\langle 1\rangle}\mathbf{c}^{\langle 2\rangle}\mathbf{a} - {}^{\langle 2\rangle}\mathbf{c}^{\langle 1\rangle}\mathbf{a})\\
&\quad - {}^{\langle 2\rangle}({}^{\langle 1\rangle}\mathbf{b}^{\langle 2\rangle}\mathbf{c} - {}^{\langle 2\rangle}\mathbf{b}^{\langle 1\rangle}\mathbf{c})^{\langle 1\rangle}({}^{\langle 1\rangle}\mathbf{c}^{\langle 2\rangle}\mathbf{a} - {}^{\langle 2\rangle}\mathbf{c}^{\langle 1\rangle}\mathbf{a})\\
&= ({}^{\langle 2\rangle}\mathbf{bc} - \mathbf{b}^{\langle 2\rangle}\mathbf{c})(\mathbf{c}^{\langle 1\rangle}\mathbf{a} - {}^{\langle 1\rangle}\mathbf{ca})\\
&\quad -(\mathbf{b}^{\langle 1\rangle}\mathbf{c} - {}^{\langle 1\rangle}\mathbf{bc})({}^{\langle 2\rangle}\mathbf{ca} - \mathbf{c}^{\langle 2\rangle}\mathbf{a})\\
&= {}^{\langle 1\rangle}\mathbf{a}^{\langle 2\rangle}\mathbf{bcc} - \mathbf{a}^{\langle 2\rangle}\mathbf{bc}^{\langle 1\rangle}\mathbf{c} - {}^{\langle 1\rangle}\mathbf{abc}^{\langle 2\rangle}\mathbf{c} + \mathbf{ab}^{\langle 1\rangle}\mathbf{c}^{\langle 2\rangle}\mathbf{c}\\
&\quad -\mathbf{ab}^{\langle 1\rangle}\mathbf{c}^{\langle 2\rangle}\mathbf{c} + {}^{\langle 2\rangle}\mathbf{abc}^{\langle 1\rangle}\mathbf{c} + \mathbf{a}^{\langle 1\rangle}\mathbf{bc}^{\langle 2\rangle}\mathbf{c} - {}^{\langle 2\rangle}\mathbf{a}^{\langle 1\rangle}\mathbf{bcc}\\
&= \mathbf{cc}({}^{\langle 1\rangle}\mathbf{a}^{\langle 2\rangle}\mathbf{b} - {}^{\langle 2\rangle}\mathbf{a}^{\langle 1\rangle}\mathbf{b}) + \mathbf{c}^{\langle 1\rangle}\mathbf{c}({}^{\langle 2\rangle}\mathbf{ab} - \mathbf{a}^{\langle 2\rangle}\mathbf{b})\\
&\quad +\mathbf{c}^{\langle 2\rangle}\mathbf{c}(\mathbf{a}^{\langle 1\rangle}\mathbf{b} - {}^{\langle 1\rangle}\mathbf{ab})\\
&= \mathbf{cc}({}^{\langle 1\rangle}\mathbf{a}^{\langle 2\rangle}\mathbf{b} - {}^{\langle 2\rangle}\mathbf{a}^{\langle 1\rangle}\mathbf{b}) + \mathbf{c}^{\langle 1\rangle}(\mathbf{c}({}^{\langle 1\rangle}\mathbf{a}^{\langle 2\rangle}\mathbf{b} -{}^{\langle 2\rangle}\mathbf{a}^{\langle 1\rangle}\mathbf{b}))
\end{aligned}$$

$$\begin{aligned} & +\mathbf{c}^{\langle 2\rangle}(\mathbf{c}({}^{\langle 1\rangle}\mathbf{a}^{\langle 2\rangle}\mathbf{b} - {}^{\langle 2\rangle}\mathbf{a}^{\langle 1\rangle}\mathbf{b})) \\ = & \ \mathbf{c}(\mathbf{c}(\mathbf{a}\times\mathbf{b})) + \mathbf{c}^{\langle 1\rangle}(\mathbf{c}(\mathbf{a}\times\mathbf{b})) + \mathbf{c}^{\langle 2\rangle}(\mathbf{c}(\mathbf{a}\times\mathbf{b})) \\ = & \ \mathbf{c}(\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})) \end{aligned}$$

Then, $(\mathbf{a}\times\mathbf{b})\cdot((\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a}))$

$$\begin{aligned} &= \sigma((\mathbf{a}\times\mathbf{b})\mathbf{c}(\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}))) \\ &= (\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}))\sigma(\mathbf{c}(\mathbf{a}\times\mathbf{b})) \\ &= (\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}))(\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})) \\ &= (\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}))^2 \end{aligned}$$

Determinants

DEFINITION 4 The alternating vector $(1,-1,1,-1,\ldots)$ for the vector space $\mathbb{R}^n$ is defined as $\aleph \equiv \sum_{i=1}^{n}(-1)^{i-1}\hat{e}_i$, where the $\hat{e}_i$ are orthonormal vectors.

THEOREM 17 If $\mathbf{a},\mathbf{b},\aleph \in \mathbb{R}^2$, then $det(\mathbf{a},\mathbf{b}) = \sigma(\aleph\,\mathbf{a}^{\langle 1\rangle}\mathbf{b})$.
Proof:

$$\begin{aligned} det(\mathbf{a},\mathbf{b}) &= a_1b_2 - a_2b_1 \\ &= \sigma((a_1,a_2)(b_2,-b_1)) \\ &= \sigma((1,-1)(a_1,a_2)(b_2,b_1)) \\ &= \sigma(\aleph\,\mathbf{a}^{\langle 1\rangle}\mathbf{b}) \end{aligned}$$

THEOREM 18 If $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{R}^3$, then
Proof:

$$\begin{aligned} det(\mathbf{a},\mathbf{b},\mathbf{c}) &= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \\ &= \sigma(\mathbf{a}(\mathbf{b}\times\mathbf{c})) \\ &= \sigma(\mathbf{a}({}^{\langle 1\rangle}\mathbf{b}^{\langle 2\rangle}\mathbf{c} - {}^{\langle 2\rangle}\mathbf{b}^{\langle 1\rangle}\mathbf{c})) \end{aligned}$$

THEOREM 19 If $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\aleph \in \mathbb{R}^4$, then

$$\begin{aligned} det(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) = \sigma(\aleph\,\mathbf{a}\,(& {}^{\langle 1\rangle}\mathbf{b}({}^{\langle 2\rangle}\mathbf{c}^{\langle 3\rangle}\mathbf{d} - {}^{\langle 3\rangle}\mathbf{c}^{\langle 2\rangle}\mathbf{d}) \\ & + {}^{\langle 2\rangle}\mathbf{b}({}^{\langle 3\rangle}\mathbf{c}^{\langle 1\rangle}\mathbf{d} - {}^{\langle 1\rangle}\mathbf{c}^{\langle 3\rangle}\mathbf{d}) \\ & + {}^{\langle 3\rangle}\mathbf{b}({}^{\langle 1\rangle}\mathbf{c}^{\langle 2\rangle}\mathbf{d} - {}^{\langle 2\rangle}\mathbf{c}^{\langle 1\rangle}\mathbf{d}))) \end{aligned}$$

THEOREM 20 If $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e} \in \mathbb{R}^5$, then

$$
\begin{aligned}
det(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}) \quad = \quad \sigma(\mathbf{a}\,(& {}^{\langle 1\rangle}\mathbf{b}({}^{\langle 2\rangle}\mathbf{c}({}^{\langle 3\rangle}\mathbf{d}^{\langle 4\rangle}\mathbf{e} - {}^{\langle 4\rangle}\mathbf{d}^{\langle 3\rangle}\mathbf{e}) \\
& \qquad + {}^{\langle 3\rangle}\mathbf{c}({}^{\langle 4\rangle}\mathbf{d}^{\langle 2\rangle}\mathbf{e} - {}^{\langle 2\rangle}\mathbf{d}^{\langle 4\rangle}\mathbf{e}) \\
& \qquad + {}^{\langle 4\rangle}\mathbf{c}({}^{\langle 2\rangle}\mathbf{d}^{\langle 3\rangle}\mathbf{e} - {}^{\langle 3\rangle}\mathbf{d}^{\langle 2\rangle}\mathbf{e})) \\
& + {}^{\langle 2\rangle}\mathbf{b}({}^{\langle 3\rangle}\mathbf{c}({}^{\langle 1\rangle}\mathbf{d}^{\langle 4\rangle}\mathbf{e} - {}^{\langle 4\rangle}\mathbf{d}^{\langle 1\rangle}\mathbf{e}) \\
& \qquad + {}^{\langle 4\rangle}\mathbf{c}({}^{\langle 3\rangle}\mathbf{d}^{\langle 1\rangle}\mathbf{e} - {}^{\langle 1\rangle}\mathbf{d}^{\langle 3\rangle}\mathbf{e}) \\
& \qquad + {}^{\langle 1\rangle}\mathbf{c}({}^{\langle 4\rangle}\mathbf{d}^{\langle 3\rangle}\mathbf{e} - {}^{\langle 3\rangle}\mathbf{d}^{\langle 4\rangle}\mathbf{e})) \\
& + {}^{\langle 3\rangle}\mathbf{b}({}^{\langle 4\rangle}\mathbf{c}({}^{\langle 1\rangle}\mathbf{d}^{\langle 2\rangle}\mathbf{e} - {}^{\langle 2\rangle}\mathbf{d}^{\langle 1\rangle}\mathbf{e}) \\
& \qquad + {}^{\langle 1\rangle}\mathbf{c}({}^{\langle 2\rangle}\mathbf{d}^{\langle 4\rangle}\mathbf{e} - {}^{\langle 4\rangle}\mathbf{d}^{\langle 2\rangle}\mathbf{e}) \\
& \qquad + {}^{\langle 2\rangle}\mathbf{c}({}^{\langle 4\rangle}\mathbf{d}^{\langle 1\rangle}\mathbf{e} - {}^{\langle 1\rangle}\mathbf{d}^{\langle 4\rangle}\mathbf{e})) \\
& + {}^{\langle 4\rangle}\mathbf{b}({}^{\langle 1\rangle}\mathbf{c}({}^{\langle 3\rangle}\mathbf{d}^{\langle 2\rangle}\mathbf{e} - {}^{\langle 2\rangle}\mathbf{d}^{\langle 3\rangle}\mathbf{e}) \\
& \qquad + {}^{\langle 2\rangle}\mathbf{c}({}^{\langle 1\rangle}\mathbf{d}^{\langle 3\rangle}\mathbf{e} - {}^{\langle 3\rangle}\mathbf{d}^{\langle 1\rangle}\mathbf{e}) \\
& \qquad + {}^{\langle 3\rangle}\mathbf{c}({}^{\langle 2\rangle}\mathbf{d}^{\langle 1\rangle}\mathbf{e} - {}^{\langle 1\rangle}\mathbf{d}^{\langle 2\rangle}\mathbf{e}))))
\end{aligned}
$$

According to *Galois* theory, roots of polynomials of degree 5 and higher cannot be characterized by closed-form solutions. Associated with each determinant is a characteristic polynomial which may reveal the connection between *Galois* theory and the determinant representation given by our vector product. A quick comparison of Theorems 19 and 20 reveals that the latter determinant cannot be expressed in terms of only cyclical and reverse-cyclical permutations of the permutation exponents in their natural, ascending order. With further study, we anticipate establishing the connection between the vector product representation and the group-theoretic results of *Galois*.

Conclusion

The utilization of this method for proving vector identities is reinforced by the very simple rules and notation. When applied to determinants, this method hints at the rudiments of *Galois* theory. Further study in this area by the authors will hopefully establish that connection in the future.