ENUMERATIVE TROPICAL ALGEBRAIC GEOMETRY

GRIGORY MIKHALKIN

Abstract. The paper establishes a formula for enumeration of curves of arbitrary genus in toric surfaces. It turns out that such curves can be counted by means of certain lattice paths in the Newton polygon. The formula was announced earlier in [17].

The result is established with the help of the so-called tropical algebraic geometry. This geometry allows to replace complex toric varieties with the real space $\mathbb{R}^n$ and holomorphic curves with certain piecewise-linear graphs there.

1. Introduction

Recall the basic enumerative problem in the plane. Let $g \geq 0$ and $d \geq 1$ be two numbers and let $\mathcal{Z} = (z_1, \dots, z_{3d-1+g})$ be a collection of points $z_j \in \mathbb{CP}^2$ in general position. A holomorphic curve $C \subset \mathbb{CP}^2$ is parameterized by a Riemann surface $\tilde{C}$ under a holomorphic map $\phi : \tilde{C} \to \mathbb{CP}^2$ so that $C = \phi(\tilde{C})$. Here we choose the minimal parametrization, i.e. such that no component of $\tilde{C}$ is mapped to a point by ϕ. The curve C is irreducible if and only if $\tilde{C}$ is connected. The number $N^{\mathrm{irr}}_{g,d}$ of irreducible curves of degree d and genus g passing through $\mathcal{Z}$ is finite and does not depend on the choice of z_j as long as this choice is generic.

Similarly we can set up the problem of counting all (not necessarily irreducible) curves. Define the genus of $C \subset \mathbb{CP}^2$ to be $\frac{1}{2}(2 - \chi(\tilde{C}))$. Note that the genus can take negative values for reducible curves. The number $N^{\mathrm{mult}}_{g,d}$ of curves of degree d and genus g passing through $\mathcal{Z}$ is again finite and does not depend on the choice of z_j as long as this choice is generic. Figure some (well-known) first few numbers $N^{\mathrm{irr}}_{g,d}$ and $N^{\mathrm{mult}}_{g,d}$.

The numbers $N^{\mathrm{irr}}_{g,d}$ are known as the Gromov-Witten invariants of $\mathbb{CP}^2$ (see [11]) while the numbers $N^{\mathrm{mult}}_{g,d}$ are sometimes called the multicomponent Gromov-Witten invariant. One series of numbers determines another by a simple combinatorial relation (see e.g. [3]). A recursive relation which allows to compute the numbers $N^{\mathrm{irr}}_{g,d}$ (and thus

The author is partially supported by the NSF.

$g\backslash d$	1	2	3	4
0	1	1	12	620
1	0	0	1	225
2	0	0	0	27
3	0	0	0	1

$g\backslash d$	1	2	3	4
-1	0	3	21	666
0	1	1	12	675
1	0	0	1	225
2	0	0	0	27

FIGURE 1. $N^{\text{irr}}_{g,d}$ and $N^{\text{mult}}_{g,d}$.

the numbers $N^{\text{mult}}_{g,d}$) was given by Kontsevich. This relation came from the associativity of the quantum cohomology (see [11]). In the arbitrary genus case Caporaso and Harris [3] gave an algorithm (bases on a degeneration of $\mathbb{CP}^2$) which allows to compute the numbers $N^{\text{mult}}_{g,d}$ (and thus the numbers $N^{\text{irr}}_{g,d}$).

The main result of this paper gives a new formula for $N^{\text{mult}}_{g,d}$. This number turns out to be the number of certain lattice paths of length $3d-1+g$ in the triangle $\Delta_d \subset \mathbb{R}^2$ with vertices $(0,0)$, $(d,0)$ and $(0,d)$. The paths have to be counted with certain non-negative multiplicities. Furthermore, this formula works not only for $\mathbb{CP}^2$ but for other toric surfaces as well. For other toric surfaces we just have to replace the triangle Δ_d with other convex lattice polygons. The polygon should be chosen so that it determines the corresponding (polarized) toric surface.

The formula comes as an application of the so-called *tropical geometry* whose objects are certain piecewise-linear polyhedral complexes in $\mathbb{R}^n$. These objects are the limits of the amoebas of holomorphic curves after a certain degeneration of the complex structure. The idea to use these objects for enumeration of holomorphic curves is due to Kontsevich.

In [12] Kontsevich and Soibelman proposed a program linking Homological Mirror Symmetry and torus fibrations from the Strominger-Yau-Zaslow conjecture. The relation is provided by passing to the so-called *"large complex limit"* which deforms a complex structure on a manifold to its worst possible degeneration.

Similar deformations appeared in other areas of mathematics under different names. The *patchworking* in Real Algebraic Geometry was discovered by Viro [23]. Maslov and his school studied the so-called dequantization of the semiring of positive real numbers (cf. [14]). The limiting semiring is isomorphic to the $(\max,+)$-semiring $\mathbb{R}_{\text{trop}}$, the semiring of real numbers equipped with taking the maximum for addition and addition for multiplication.

The semiring $\mathbb{R}_{\text{trop}}$ is known to computer scientists as one of *tropical* semirings, see e.g. [19]. In Mathematics this semiring appears from some non-Archimedean fields K under a certain pushing forward to $\mathbb{R}$ of the arithmetic operations in K.

In this paper we develop some basic algebraic geometry over $\mathbb{R}_{\text{trop}}$ with a view towards counting curves. In particular, we rigorously setup some enumerative problems over $\mathbb{R}_{\text{trop}}$ and prove their equivalence to the relevant problems of complex and real algebraic geometry. The reader can refer to Chapter 9 of Sturmfels' recent book [22] for some first steps in tropical algebraic geometry.

We solve the corresponding tropical enumerative problem in $\mathbb{R}^2$. As an application we get a formula counting the number of curves of given degree and genus in terms of certain lattice paths of a given length in the relevant Newton polygon. In particular this gives an interpretation of the Gromov-Witten invariants in $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ via lattice paths in the triangle and the rectangle respectively. This formula was announced in [17]. For the proof we use the patchworking side of the story which is possible to use since the ambient space is 2-dimensional and the curves there are hypersurfaces. An alternative approach (applicable to higher dimensions as well) is to use the symplectic field theory of Eliashberg, Givental and Hofer [4]. Generalization of this formula to higher dimensions is a work in progress. In this paper we only define the enumerative multiplicity for the 2-dimensional case (in higher dimensions it is not localized at the vertices).

The main theorems are stated in section 7 and proved in section 8. In section 2 we define tropical curves geometrically (in a way similar to webs of Aharony, Hanany and Kol [1], [2]). In section 3 we exhibit them as algebraic objects over the tropical semifield. In section 4 we define the tropical enumerative problems in $\mathbb{R}^2$, in section 5 recall those in $(\mathbb{C}^*)^2$. Section 7 is auxiliary for section 8 and deals with certain piecewise-holomorphic piecewise-Lagrangian objects in $(\mathbb{C}^*)^2$ called *complex tropical curves*.

The author is grateful to Y. Eliashberg, K. Hori, I. Itenberg, M. Kapranov, M. Kontsevich, A. Okounkov, B. Sturmfels, R. Vakil, O. Viro and J.-Y. Welschinger for useful discussions.

2. Geometry of tropical curves

In this section we define tropical curves in $\mathbb{R}^n$ *geometrically* and set up the corresponding enumerative problem. The following section will exhibit such curves as one-dimensional varieties over the so-called *tropical* semiring.

2.1. **Definitions and the first examples.** Let $\bar{\Gamma}$ be a *weighted* finite graph. The weights are natural numbers prescribed to the edges.

Clearly, $\bar{\Gamma}$ is a compact topological space. We make it non compact by removing the set of all 1-valent vertices $\mathcal{V}_1$,

$$\Gamma = \bar{\Gamma} \smallsetminus \mathcal{V}_1.$$

Definition 2.1. A proper map $h : \Gamma \to \mathbb{R}^n$ is called *a parameterized tropical curve* if it satisfies to the following two conditions.

- For every edge $E \subset \Gamma$ the restriction $h|_E$ is an embedding. The image $h(E)$ is contained in a line $l \subset \mathbb{R}^2$ such that the slope of l is rational.
- For every vertex $V \in \Gamma$ we have the following property. Let $E_1, \dots, E_n \subset \Gamma$ be the edges adjacent to V, let $w_1, \dots, w_n \in \mathbb{N}$ be their weights and let $v_1, \dots, v_n \in \mathbb{Z}^n$ be the primitive integer vectors from V in the direction of the edges. We have

$$\sum_{j=1}^{n} w_j v_j = 0. \tag{1}$$

We say that two parameterized tropical curves $h : \Gamma \to \mathbb{R}^n$ and $h' : \Gamma' \to \mathbb{R}^n$ are *equivalent* if there exists a homeomorphism $\Phi : \Gamma \to \Gamma'$ which respects the weights of the edges and such that $h = h' \circ \Phi$. In this paper we do not distinguish equivalent parameterized tropical curves.

The image

$$C = h(\Gamma) \subset \mathbb{R}^n$$

is called the (unparameterized) tropical curve. It is a weighted piecewise-linear graph in $\mathbb{R}^n$.

Remark 2.2. In dimension 2 the notion of tropical curve coincides with the notion of (p, q)-webs introduced by Aharony, Hanany and Kol in [2] (see also [1]).

Example 2.3. Consider the union of three simple rays

$$Y = \{(x, 0) \mid x \leq 0\} \cup \{(0, y) \mid y \leq 0\} \cup \{(x, x) \mid x \geq 0\}.$$

This graph is a tropical curve since $(-1, 0) + (0, -1) + (1, 1) = 0$. A parallel translation of Y in any direction in $\mathbb{R}^2$ is clearly also a tropical curve. This gives us a 2-dimensional family of curves in $\mathbb{R}^2$. Such curves are called *tropical lines*.

Remark 2.4. The term *tropical lines* is justified in the next section dealing with the underlying algebra. So far we would like to note the following properties of this family, see Figure 2.

- For any two points in $\mathbb{R}^2$ there is a tropical line passing through them.
- Such a line is unique if the choice of these two points is generic.

- Two generic tropical lines intersect in a single point.

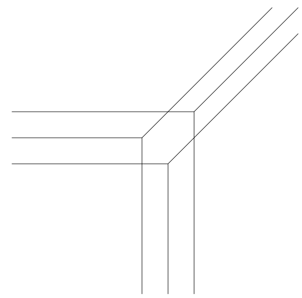

FIGURE 2. Three distinct tropical lines.

Somewhat more complicated tropical curves (corresponding to projective curves of degree 3) are pictured on Figure 3

2.2. **The degree of a tropical curve.** Let $\mathcal{T} = \{\tau_1, \dots, \tau_q\} \subset \mathbb{Z}^n$ be a set of non-zero integer vectors such that $\sum_{j=1}^{q} \tau_j = 0$. Suppose that in this set we do not have positive multiples of each other, i.e. if $\tau_j = m\tau_k$ for $m \in \mathbb{N}$ then $\tau_j = \tau_k$. The degree of a tropical curve $C \subset \mathbb{R}^n$ takes values in such sets according to the following construction.

Note that a tropical curve $C \subset \mathbb{R}^n$ has a finite number of ends, i.e. unbounded edges (rays). Let $\tau \in \mathbb{Z}^n$ be a primitive vector. Its positive multiple is included in $\mathcal{T}$ if and only if there exist the ends of C in the direction of τ. In such case we include $m\tau$ into $\mathcal{T}$, where m is the sum of multiplicities of all such rays.

Definition 2.5. The resulting set $\mathcal{T}$ is called *the toric degree* of C.

Note that the sum of all vectors in $\mathcal{T}$ is zero. This follows from adding the conditions (1) from Definition 2.1 in all vertices of C.

For example the degree of both curves from Figure 5 is $\{(-1,-1), (2,-1), (-1,2)\}$ while the degree of both curves from Figure 3 is $\{(-3,0),(0,-3),(3,3)\}$.

Definition 2.6. If the toric degree of a tropical curve $C \subset \mathbb{R}^n$ is $\{(-d,0),(0,-d),(d,d)\}$ then C is called a tropical *projective curve of degree d*.

The curves from Figure 3 are examples of planar projective cubics.

2.3. **Genus of a tropical curve and the Riemann-Roch formula.** We say that a tropical curve is *reducible* if it can be parameterized by a disconnected graph. Otherwise we say that it is *irreducible*.

Definition 2.7. *The genus of a parameterized tropical curve* $\Gamma \to \mathbb{R}^n$ is $\dim H_1(\Gamma) - \dim H_0(\Gamma) + 1$. In particular, if Γ is connected then its genus is the first Betti number of Γ. *The genus of a tropical curve* $C \subset \mathbb{R}^n$ is the minimum genus among all parameterizations of C.

Note that according to this definition the genus can be negative for reducible curves. E.g. the genus of the union of three lines from Figure 2 has genus -2.

If $C \subset \mathbb{R}^n$ is an embedded 3-valent graph then the parameterization is unique and is an embedding. However, in general, there might be several parameterizations of different genus and taking the minimal value is essential.

Example 2.8. The curve on the right-hand side of Figure 3 can be parameterized by a tree once we "resolve" its 4-valent vertex. Therefore, its genus is 0.

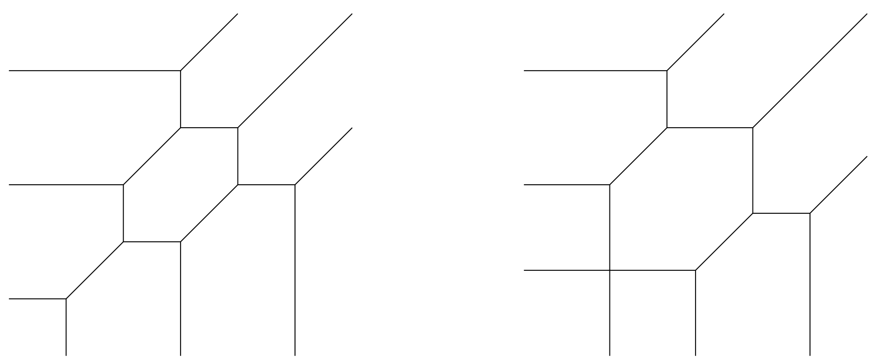

FIGURE 3. A smooth projective tropical cubic and a rational (genus 0) projective tropical cubic.

As in the classical complex geometry case the tropical curves are subject to the constraint of *the Riemann-Roch formula.* Let x be the number of ends of a tropical curve $C \subset \mathbb{R}^n$. This number can be considered as a tropical counterpart of minus the value of the canonical class of the ambient complex variety on the curve.

Let us consider the space of deformation of a parameterized tropical curve $h : \Gamma \to \mathbb{R}^n$. There are two things that we can deform, the map h or the graph Γ itself. The second type of deformation can only arise if Γ has vertices of valence 4 or more.

Recall that (as in Chemistry) the *valence* of a vertex of a weighted graph is the sum of multiplicities of the adjacent edges.

Definition 2.9. A parameterized tropical curve $h : \Gamma \to \mathbb{R}^n$ is called *simple* if

- Γ is 3-valent,
- h is an immersion,
- if $a, b \in \Gamma$ are such that $h(a) = h(b)$ then neither a nor b can be a vertex of Γ.

In this case the image $h(\Gamma)$ is called a *a simple tropical curve*.

Note that a 3-valent graph parameterizing a tropical curve in $\mathbb{R}^n$, $n \geq 1$, has all its edges of weight 1. A simple parameterization of C (if it exists) is always of minimal genus. Furthermore, any other parameterization has a higher genus.

We are ready to formulate the tropical Riemann-Roch theorem for simple curves.

Theorem 2.10. *Suppose that Γ is a 3-valent graph that parameterizes a tropical curve $h : \Gamma \to \mathbb{R}^n$. Then h locally varies in a (real) linear k-dimensional space, where*

$$k \geq x + (n-3)(1-g).$$

Proof. Let $\Gamma' \subset \Gamma$ be a finite tree containing all the vertices of Γ. Note that the number of finite edges in $\Gamma \smallsetminus \Gamma'$ is g. By an Euler characteristic computation we get that the number of finite edges of Γ' is equal to $x - 3 + 2g$.

Maps $\Gamma' \to \mathbb{R}^n$ vary in a linear $(x-3+2g+n)$-dimensional family if we do not change the slopes of the edges. The $(x-3+2g)$-dimensional part comes from varying of the lengths of the edges while the n-dimensional part comes from translations in $\mathbb{R}^n$. Such a map is extendable to a tropical map $\Gamma \to \mathbb{R}^n$ if the pairs of vertices corresponding to the g remaining edges define the lines with the correct slope. This each of the g edges imposes a linear condition of codimension at most $n-1$. Thus tropical perturbations of $\Gamma \to \mathbb{R}^n$ form a linear family of dimension at least $x - 3 + 2g + n - (n-1)g = x + (n-3)(1-g)$. □

Corollary 2.11. *A simple tropical curve $C \subset \mathbb{R}^n$ locally varies in a (real) linear k-dimensional space, where*

$$k \geq x + (n-3)(1-g).$$

Consider the general case now and suppose that Γ has vertices of valence higher than 3. How much Γ differs from a 3-valent graph is measured by the following invariant. Let the *overvalence* $\mu(\Gamma)$ be the sum of the valences of all vertices of valence higher than 3 minus the number of such vertices. Thus $\mu(\Gamma) = 0$ if and only if no vertex of Γ has valence higher than 3.

We can deform Γ to a 3-valent graph by the following procedure reducing μ. If we have $n > 3$ edges adjacent to the same vertex then we can separate them into two groups so that each group contains at least 2 edges. Let us insert a new edge E' separating these groups as shown in Figure 4. This replaces the initial n-valent vertex with 2 vertices (the endpoints of E') of smaller valence. There is a "virtual slope" of E' determined by the slopes of the edges in each group. This slope appears in local perturbation of the tropical map $\Gamma \to \mathbb{R}^n$ corresponding to this modification of Γ. We say that Γ_j is a *partial smoothing* of Γ if Γ_j is

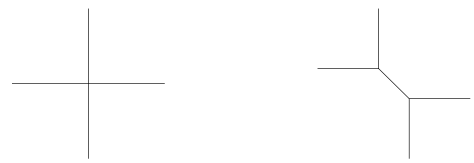

FIGURE 4. Smoothing a vertex of higher valence.

obtained by a sequence of such procedures starting from Γ.

Theorem 2.10 can be generalizes in the following way.

Theorem 2.12. *The space of deformations of a parameterized tropical curve $h : \Gamma \to \mathbb{R}^n$ is locally a cone*

$$\mathcal{C} = \bigcup_j \mathcal{C}_j,$$

where $\mathcal{C}_j$ is a convex cone corresponding to tropical curves parameterized by partial smoothings Γ_j of Γ. The union is taken over all possible partial smoothings Γ_j. Unless $\mathcal{C}_j$ is empty we have

$$\dim \mathcal{C}_j \geq x + (n-3)(1-g) - \mu(\Gamma_j).$$

Proof. The proof of Theorem 2.10 is applicable here as well. But one has to note that $\mu(\Gamma_j)$ of the finite length edges of the maximal tree Γ'_j have zero length and therefore cannot vary. □

Remark 2.13. Curves which vary in a family strictly larger than $x + (n-3)(1-g)$ (cf. Theorem 2.10) are called *superabundant*. In contrast to the classical case superabundancy can be easily seen geometrically. This is the case if some of the cycles of $C \subset \mathbb{R}^n$ are contained in smaller-dimensional affine-linear subspaces of $\mathbb{R}^n$, e.g. if a spatial curve develops a planar cycle. Dimension 2 is the smallest dimension where one can have a cycle of such type in a genus-minimizing parameterization so superabundancy can only appear in dimension 3 or higher. Thus one may strengthen Theorem 2.10 in the case $n = 2$ (or $g = 0$).

Theorem 2.14. *A simple parameterized tropical curve $h : \Gamma \to \mathbb{R}^n$ locally varies in a (real) linear k-dimensional space, where*

$$k = x + (n - 3)(1 - g)$$

if either $n = 2$ or $g = 0$.

This theorem also follows from the proof of Theorem 2.10 once we note that the edges of $\Gamma \smallsetminus \Gamma'$ pose independent conditions in this case.

2.4. **Smooth vs. singular curves, multiplicity of a vertex.** Suppose that a parameterized tropical curve $h : \Gamma \to \mathbb{R}^n$ is simple, so there is no way to smooth Γ by reducing the valence of its vertices. Even in such case the (unparameterized) curve $h(\Gamma)$ can be deformed with a change in the combinatorial type of Γ.

Example 2.15. Let Γ be the union of 3 rays in $\mathbb{R}^2$ in the direction $(-2, 1)$, $(1, -2)$ and $(1, 1)$ emanating from the origin (pictured on the left-hand side of Figure 5). This curve is a simple tropical curve of genus 0.

It can be obtained as a $t = 0$ limit of the family of genus 1 curves given by the union of 3 rays in $\mathbb{R}^2$ in the direction $(-2, 1)$, $(1, -2)$ and $(1, 1)$ emanating from and $(-2t, t)$, $(t, -2t)$ and (t, t) respectively and the three intervals $[(-2t, t), (t, -2t)]$, $[(-2t, t), (t, t)]$ and $[(t, t), (t, -2t)]$ as pictured in Figure 5.

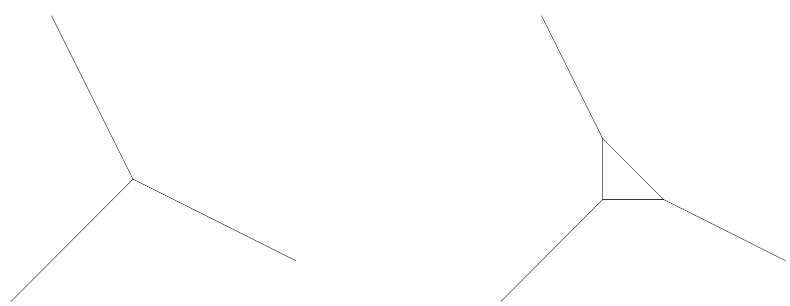

FIGURE 5. Perturbation at a non-smooth 3-valent vertex

Let $C = h(\Gamma)$ be a simple curve. By a vertex V of C we mean the image of a (3-valent) vertex of Γ. As in Definition 2.1 let $w_1, \dots, w_3$ be their weights of the edges adjacent to V and let $v_1, \dots, v_3$ be the primitive integer vectors in the direction of the edges.

Definition 2.16. The *multiplicity* of C at its 3-valent vertex V is $w_1w_2|v_1 \times v_2|$. Here $|v_1 \times v_2|$ is the length of the vector product of v_1 and v_2 or, alternatively, the area of the parallelogram spanned by v_1 and v_2. Note that

$$w_1w_2|v_1 \times v_2| = w_2w_3|v_2 \times v_3| = w_3w_1|v_3 \times v_1|$$

since $v_1w_1 + v_2w_2 + v_3w_3 = 0$ by Definition 2.1.

Definition 2.17. A tropical curve $C \subset \mathbb{R}^n$ is called *smooth* if it can be parameterized by $h : \Gamma \to \mathbb{R}^n$ so that Γ is 3-valent, h is an embedding and the multiplicity of every vertex of C is 1. A tropical curve is called *smoothly immersed* if it is a simple curve of multiplicity 1.

3. Underlying tropical algebra

In this section we exhibit the tropical curves as algebraic varieties with respect to a certain algebra and also define higher-dimensional tropical algebraic varieties in $\mathbb{R}^n$.

3.1. **The tropical semifield $\mathbb{R}_{\text{trop}}$.** Consider the semiring $\mathbb{R}_{\text{trop}}$ of real numbers equipped with the following arithmetic operations called *tropical* in Computer Science:

$$\text{“}x + y\text{”} = \max\{x, y\} \qquad \text{“}xy\text{”} = x + y,$$

$x, y \in \mathbb{R}_{\text{trop}}$. We use the quotation marks to distinguish the tropical operations from the classical ones. Note that addition is idempotent, “$x + x = x$”. This makes $\mathbb{R}_{\text{trop}}$ to a semiring without the additive zero (the role of such zero would be played by $-\infty$).

Remark 3.1. According to [19] the term “tropical” appeared in Computer Science in honor of Brazil and, more specifically, after Imre Simon (who is a Brazilian computer scientist) by Dominique Perrin. In Computer Science the term is usually applied to $(\min, +)$ semirings. Our semiring $\mathbb{R}_{\text{trop}}$ is $(\max, +)$ by our definition but isomorphic to the $(\min, +)$-semiring, the isomorphism is given by $x \mapsto -x$.

As usual, a (Laurent) polynomial in n variables over $\mathbb{R}_{\text{trop}}$ is defined by

$$f(x) = \text{“}\sum_{j\in A} a_j x^j\text{”} = \max_{j\in A}(< j, x > +a_j),$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $j = (j_1, \dots, j_n)$, $x^j = x_1^{j_1} \dots x_n^{j_n}$, $< j, x > = j_1x_1 + \dots + j_nx^n$ and $A \subset \mathbb{Z}^n$ is a finite set. Note that $f : \mathbb{R}^n \to \mathbb{R}$ is a convex piecewise-linear function. It coincides with the Legendre transform of a function $j \mapsto -a_j$ defined on the finite set A.

Definition 3.2. The polyhedron $\Delta = \text{ConvexHull}(A)$ is called the *Newton polyhedron* of f. It can be treated as a refined version of the degree of the polynomial f.

3.2. **Tropical hypersurfaces: the variety of a tropical polynomial.** For a tropical polynomial f in n variables we define its *variety* $V_f \subset \mathbb{R}^n$ as the set of points where the piecewise-linear function f is not smooth, cf. [9], [17] and [22]. In other words, V_f is the corner locus of f.

Proposition 3.3. *V_f is the set of points in $\mathbb{R}^n$ where more than one monomial of f reaches its maximal value.*

Proof. If exactly one monomial of $f(x) = \text{``}\sum_{j\in A} a_j x^j\text{''} = \max_{j\in A}(< j, x > +a_j)$ is maximal at $x \in \mathbb{R}^n$ then f locally coincides with this monomial and, therefore, linear and smooth. Otherwise f has a corner at x. □

Remark 3.4. At a first glance this definition might appear to be unrelated to the classical definition of the variety as the zero locus of a polynomial. To see a connection recall that there is no additive zero in $\mathbb{R}_{\text{trop}}$, but its rôle is played bu $-\infty$.

Consider the graph $\Gamma_f \subset \mathbb{R}^n \times \mathbb{R}$ of a tropical polynomial $f : \mathbb{R}^n \to \mathbb{R}$. The graph Γ_f itself is not a tropical variety in $\mathbb{R}^{n+1}$ but it can be completed to the tropical variety

$$\bar{\Gamma}_f = \Gamma_f \cup \{(x, y) \mid x \in V_f,\ y \leq f(x)\}.$$

Proposition 3.5. *$\bar{\Gamma}_f$ coincides with the variety of the polynomial in $(n+1)$ variables $y + f(x)$, $y \in \mathbb{R}$, $x \in \mathbb{R}^n$.*

Proof. If $(x, y) \in \Gamma_f$ then we have y and one of the monomials of f both reaching the maximal values in "$y + f(x)$" $= \max\{y, f(x)\}$. If $x \in V_f$ and $y < f(x)$ then two monomials of $f(x)$ are reaching the maximal value at the expression "$y + f(x)$". □

Note that we have $V_f = \bar{\Gamma}_f \cap \{y = t\}$ for t sufficiently close to $-\infty$. This is the sense in which V_f can be thought as a zero locus.

One may argue that Γ_f itself is a *subtropical* variety (as in subanalytic vs. analytic sets) while $\bar{\Gamma}_f$ is its tropical closure. Figure 6 sketches the graph $y =$ "$ax^2 + bx + c$" and its tropical closure.

Definition 3.6. Varieties $V_f \subset \mathbb{R}^n$ are *tropical hypersurfaces.*

Remark 3.7. Different tropical polynomials may define the same varieties. Recall that a function $\phi : A \to \mathbb{R}$ is called *concave* if for any

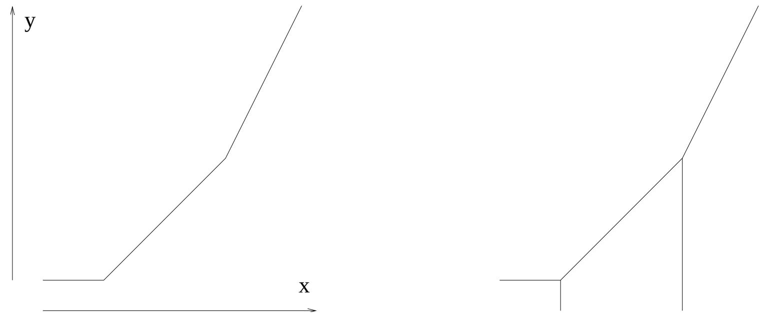

FIGURE 6. The graph $y =$ “ax^2+bx+c” and its closure, the tropical parabola

(possibly non-distinct) $b_0, \dots, b_n \in A \subset \mathbb{R}^n$ and any $t_0, \dots, t_n \geq 0$ with $\sum_{k=0}^{n} t_k = 1$ we have

$$\phi(\sum_{k=0}^{n} t_k b_k) \geq \sum_{k=0}^{n} t_k \phi(b_k).$$

We have three types of ambiguities when $f \neq g$ but $V_f = V_g$.

- $g =$ “$x_j f$”, where z_j is a coordinate in $\mathbb{R}^n_{\text{trop}}$. Note that in this case the Newton polyhedron of g is a translate of the Newton polyhedron of f.
- $g =$ “cf”, where $c \in \mathbb{R}_{\text{trop}}$ is a constant.
- The function $\Delta \cap \mathbb{Z}^n \ni j \mapsto a_j$ is not concave, where $f =$ “$\sum_{j \in A} a_j x^j$” and we set $a_j \mapsto -\infty$ if $j \notin A$. Then the variety of f coincides with the variety of g where g is the smallest concave function such that $g \geq f$ (in other words g is a concave hull of f).

Thus to define tropical hypersurfaces it suffices to consider only tropical polynomials whose coefficients satisfy to the concavity condition above.

Proposition 3.8 ([17])**.** *The space of all tropical hypersurfaces with a given Newton polydedron Δ is a closed convex polyhedral cone $\mathcal{M}_\Delta \subset \mathbb{R}^m$, $m = \#(\Delta \cap \mathbb{Z}^n) - 1$. The cone $\mathcal{M}_\Delta \subset \mathbb{R}^m$ is well-defined up to the natural action of $SL_m(\mathbb{Z})$.*

Proof. All convex functions $\Delta \cap \mathbb{Z}^n \to \mathbb{R}, j \mapsto a_j$ form a closed convex polyhedral cone $\tilde{\mathcal{M}}_\Delta \subset \mathbb{R}^{m+1}$. But the function $j \mapsto a_j + c$ defines the same curve as the function $j \mapsto a_j$. To get rid of this ambiguity we choose $j' \in \Delta \cap \mathbb{Z}^2$ and define $\mathcal{M}_\Delta$ as the image of $\tilde{\mathcal{M}}_\Delta$ under the linear projection $\mathbb{R}^{m+1} \to \mathbb{R}^m, a_j \mapsto a_j - a_{j'}$. $\square$

3.3. **Compactness of the space of tropical hypersurfaces.** Clearly, the cone $\mathcal{M}_\Delta$ is not compact. Nevertheless it gets compactified by the cones $\mathcal{M}_{\Delta'}$ for all non-empty lattice subpolyhedra $\Delta' \subset \Delta$ (including polygons with the empty interior). Indeed, we have the following proposition.

Proposition 3.9. *Let $C_k \subset \mathbb{R}^n$, $k \in \mathbb{N}$ be a sequence of tropical hypersurfaces whose Newton polyhedron is Δ. There exists a subsequence which converges to a tropical curve C whose Newton polyhedron Δ_C is contained in Δ (note that C is empty if Δ_C is a point). The convergence is in the Hausdorff metric when restricted to any compact subset in $\mathbb{R}^n$. Furthermore, if the Newton polyhedron of C coincides with Δ then the convergence is in the Hausdorff metric in the whole $\mathbb{R}^n$.*

Proof. Each C_n is defined by a tropical polynomial $f^{C_n}(x) = \text{“}\sum_j a_j^{C_n} x^j\text{”}$. We may assume that the coefficients $a_j^{C_n}$ are chosen so that that they satisfy to the concavity condition and so that $\max_j a_j^{C_n} = 0$. This takes care of the ambiguity in the choice of f^{C_n} (since the Newton polyhedron is already fixed).

Passing to a subsequence we may assume that $a_j^{C_n}$ converge (to a finite number or $-\infty$) when $n \to \infty$ for all $j \in \Delta \cap \mathbb{Z}^n$. By our assumption one of these limits is 0. Define C to be the variety of “$\sum a_j^\infty x^j$”, where we take only finite coefficients $a_j^\infty = \lim_{n\to\infty} a_j^n > -\infty$. □

3.4. **Lattice subdivision of Δ associated to a tropical hypersurface.** A tropical polynomial f defines a lattice subdivision of its Newton polyhedron Δ in the following way (cf. [6]). Define the (unbounded) *extended polyhedral domain*

$$\tilde{\Delta} = \mathrm{ConvexHull}\{(j,t) \mid j \in A, t \leq a_j\} \subset \mathbb{R}^n \times \mathbb{R}.$$

The projection $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ induces a homeomorphism from the union of all closed bounded faces of $\tilde{\Delta}$ to Δ.

Definition 3.10. The resulting lattice subdivision Subdiv_f of Δ is called the *subdivision associated to f*.

Proposition 3.11. *The lattice subdivision* Subdiv_f *is dual to the tropical hypersurface V_f. Namely, for every k-dimensional polygon $\Delta' \in$* Subdiv_f *such that $\Delta' \not\subset \partial\Delta$ there is a convex polyhedron $V_f^{\Delta'} \subset V_f$. This correspondence has the following properties.*

- *$V_f^{\Delta'}$ is contained in an $(n-k)$-dimensional affine-linear subspace $L^{\Delta'}$ of $\mathbb{R}^n$ orthogonal to Δ'.*

- *The relative interior* $U_f^{\Delta'}$ *of* $V_f^{\Delta'}$ *in* $L^{\Delta'}$ *is not empty.*
- $V_f = \bigcup U_f^{\Delta'}$.
- $U_f^{\Delta'} \cap U_f^{\Delta''} = \emptyset$ *if* $\Delta' \neq \Delta''$.

Proof. For every $\Delta' \in \mathrm{Subdiv}_f$, $\Delta' \not\subset \partial\Delta$ consider the *truncated* polynomial

$$f^{\Delta'}(x) = \sum_{j\in\Delta'} a_j x^j$$

(recall that $f(x) = \sum_{j\in\Delta} a_j x^j$). Define

$$V_f^{\Delta'} = V_f \cap \bigcap_{\Delta''\subset\Delta'} V_{f^{\Delta''}}. \tag{2}$$

Note that for any face $\Delta'' \subset \Delta'$ we have the variety $V_{f^{\Delta''}}$ orthogonal to Δ'' and therefore to Δ'. To see that $V^{\Delta'}$ is compact we restate the defining equation (2) algebraically: $V^{\Delta'}$ is the set of points where all monomials of f indexed by Δ' have equal values while the value of any other monomial of f could only be smaller. □

Example 3.12. Figure 7 shows the subdivisions dual to the curves from Figure 3 and 5.

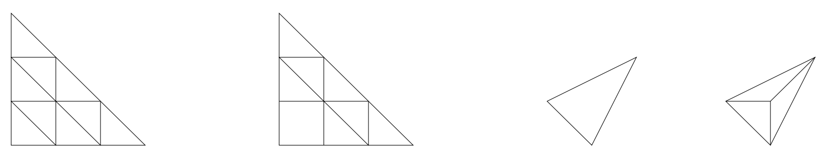

FIGURE 7. Lattice subdivisions associated to the curves from Figure 3 and Figure 5.

It was observed in [9], [17] and [22] that V_f is an $(n-1)$-dimensional polyhedral complex dual to the subdivision Subdiv_f. The complex $V_f \subset \mathbb{R}^n$ is a union of convex (not necessarily bounded) polyhedra or *cells* of V_f. Each k-cell (even if it is unbounded) of V_f is dual to a bounded $(n-k-1)$-face of $\tilde{\Delta}$, i.e. to an $(n-k-1)$-cell of Subdiv_f. In particular, the slope of each cell of V_f is rational.

In particular, a $(n-1)$ dimensional cell is dual to an interval $I \subset \mathbb{R}^n$ whose both ends are lattice points. We define the *lattice length* of I as $\#(I \cap \mathbb{Z}^n) - 1$. (Such length is invariant with respect to $SL_n(\mathbb{Z})$.) We can treat V_f as a weighted piecewise-linear polyhedral complex in $\mathbb{R}^n$, the weights are natural numbers associated to the $(n-1)$-cells. They are the lattice lengths of the dual intervals.

Definition 3.13. The *combinatorial type* of a tropical hypersurface $V_f \subset \mathbb{R}^n$ is the equivalence class of all V_g such that $\mathrm{Subdiv}_g = \mathrm{Subdiv}_f$.

Let $\mathcal{S}$ be such a combinatorial type.

Lemma 3.14. *All tropical hypersurfaces of the same combinatorial type $\mathcal{S}$ form a convex polyhedral domain $\mathcal{M}_\mathcal{S} \subset \mathcal{M}_\Delta$ that is open in its affine-linear span.*

Proof. The condition $\mathrm{Subdiv}_f = \mathcal{S}$ can be written in the following way in terms of the coefficients of $f(x) =$ "$\sum_j a_j x^j$". For every $\Delta' \in \mathcal{S}$ the function $j \mapsto -a_j$ for $j \in \Delta'$ should coincide with some linear function $\alpha : \mathbb{Z}^n \to \mathbb{R}$ such that $-a_j > \alpha(j)$ for every $j \in \Delta \smallsetminus \Delta'$. □

It turns out that the weighted piecewise-linear complex V_f satisfies the balancing property at each $(n-2)$-cell, see Definition 3 of [17]. Namely, let $F_1, \dots, F_k$ be the $(n-1)$-cells adjacent to a $(n-2)$-cell G of V_f. Each F_j has a rational slope and is assigned a weight w_j. Choose a direction of rotation around G and let $c_{F_j} : \mathbb{Z}^n \to \mathbb{Z}$ be linear maps whose kernels are planes parallel to F_j and such that they are primitive (non-divisible) and agree with the chosen direction of rotation. The balancing condition states that

$$\sum_{j=1}^{k} w_j c_{F_j} = 0. \tag{3}$$

As it was shown in [17] this balancing condition at every $(n-2)$-cell of a rational piecewise-linear $(n-1)$-dimensional polyhedral complex in $\mathbb{R}^n$ suffices for such a polyhedral complex to be the variety of some tropical polynomial.

Theorem 3.15 ([17])**.** *A weighted $(n-1)$-dimensional polyhedral complex $\Pi \subset \mathbb{R}^n$ is the variety of a tropical polynomial if and only if each k-cell of Π is a convex polyhedron sitting in a k-dimensional affine subspace of $\mathbb{R}^n$ with a rational slope and Π satisfies to the balancing condition* (3) *at each $(n-2)$-cell.*

This theorem implies that the definitions of tropical curves and tropical hypersurfaces agree.

Corollary 3.16. *Any tropical curve $C \subset \mathbb{R}^2$ is a tropical hypersurface for some polynomial f. Conversely, any tropical hypersurface in $\mathbb{R}^2$ can be parameterized by a tropical curve.*

Remark 3.17. Furthermore, the degree of C is determined by the Newton polygon Δ of f according to the following recipe. If $[a, b] \subset \mathbb{R}^n$

is an interval with $a, b \in \mathbb{Z}^n$ let us denote with $|[a,b]| = \max\{k \in \mathbb{N} \mid \frac{1}{k}(b-a) \in \mathbb{Z}^n\}$ its *integer length*. For each side $\Delta' \subset \partial\Delta$ we take the primitive integer outward normal vector $|\Delta'|$ times to get the degree of C.

3.5. **Tropical varieties and non-Archimendian amoebas.** Polyhedral complexes resulting from tropical varieties appeared in [9] in the following context. Let K be a *complete algebraically closed non-Archimedian field*. This means that K is an algebraically closed field and there is a *valuation* $\mathrm{val} : K^* \to \mathbb{R}$ defined on $K^* = K \smallsetminus \{0\}$ such that e^{val} defines a complete metric on K. Recall that a valuation val is a map such that $\mathrm{val}(xy) = \mathrm{val}(x) + \mathrm{val}(y)$ and $\mathrm{val}(x+y) \leq \max\{\mathrm{val}(x), \mathrm{val}(y)\}$.

Our principal example of such K is a field of *Puiseux series* with real powers. To construct K we take the algebraic closure $\bar{\mathbb{C}}((t))$ of the field of Laurent series $\mathbb{C}((t))$. The elements of $\bar{\mathbb{C}}((t))$ are formal power series in t $a(t) = \sum\limits_{k \in A} a_k t^k$, where $a_k \in \mathbb{C}$ and $A \subset \mathbb{Q}$ is a subset bounded from below and contained in an arithmetic progression. We set $\mathrm{val}(a(t)) = -\min A$. We define K to be the completion of $\bar{\mathbb{C}}((t))$ as the metric space with respect to the norm e^{val}.

Let $V \subset (K^*)^n$ be an algebraic variety over K. The image of V under the map $\mathrm{Val} : (K^*)^n \to \mathbb{R}^n$, $(z_1, \dots, z_n) \mapsto (\mathrm{val}(z_1), \dots, \mathrm{val}(z_n))$ is called *the amoeba of* V (cf. [6]). Kapranov [9] has shown that the amoeba of a non-Archimedian hypersurface is the variety of a tropical polynomial. Namely, if $\sum\limits_{j \in A} a_j z^j = 0$, $0 \neq a_j \in K$ is a hypersurface in $(K^*)^n$ then its amoeba is the variety of the tropical polynomial $\sum\limits_{j \in A} \mathrm{val}(a_j) x^j$.

More generally, if F is a field with a real-valued norm then the amoeba of an algebraic variety $V \subset (F^*)$ is $\mathrm{Log}(V) \subset \mathbb{R}^n$, where $\mathrm{Log}(z_1, \dots, z_n) = (\log ||z_1||, \dots, \log ||z_n||)$. Note that Val is such map with respect to the non-Archimedian norm e^{val} in K.

Another particularly interesting case is if $F = \mathbb{C}$ with the standard norm $||z|| = z\bar{z}$ (see [6],[15],[18], etc.). The non-Archimedian hypersurface amoebas appear as limits in the Hausdorff metric of $\mathbb{R}^n$ from the complex hypersurfaces amoebas (see e.g. [16]).

It was noted in [21] that the non-Archimedian approach can be used for to define tropical varieties of arbitrary codimension in $\mathbb{R}^n$. Namely, one can define the tropical varieties in $\mathbb{R}^n$ to be the images $\mathrm{Val}(V)$ of arbitrary algebraic varieties $V \in (K^*)^n$. This definition allow to avoid dealing with the intersections of tropical hypersurfaces in non-general position. We refer to [21] for relevant discussions.

4. Enumeration of tropical curves in $\mathbb{R}^2$

4.1. **Simple curves and their lattice subdivisions.** Corollary 3.16 implies that any tropical curve is a tropical hypersurface, i.e. it is the variety of a tropical polynomial $f : \mathbb{R}^2 \to \mathbb{R}$. By Remark 3.7 the Newton polygon Δ of such f is well-defined up to a translation.

Definition 4.1. We call Δ *the degree of a tropical curve in* $\mathbb{R}^2$.

By Remark 3.17 this degree supplies the same amount of information as the toric degree from Definition 2.5. We extract two numerical characteristics from the polygon $\Delta \subset \mathbb{R}^2$:

$$s = \#(\partial\Delta \cap \mathbb{Z}^2),\ l = \#(\operatorname{Int}\Delta \cap \mathbb{Z}^2). \tag{4}$$

It is convenient also to consider $m = \#(\Delta \cap \mathbb{Z}^2) - 1 = s + l - 1$. The number m is the dimension of the space of tropical curves of degree Δ (since $m+1$ is the number of coefficients in the corresponding polynomials). The number s is the number of unbounded edges of the curve if counted with multiplicities (recall that we denoted the number of unbounded edges counted simply with $x \leq s$). The number l is the genus of a smooth curve of degree Δ. To see this let us note that every lattice point of Δ is a vertex of the associated subdivision for a smooth curve C. Therefore, the homotopy type of C coincides with $\operatorname{Int}\Delta \smallsetminus \mathbb{Z}^2$. Note also that smooth curves are dense in $\mathcal{M}_\Delta$.

Lemma 4.2. *A tropical curve $C \subset \mathbb{R}^2$ is simple (see Definition 2.9) if and only if it is the variety of a tropical polynomial such that* Subdiv_f *is a subdivision into triangles and parallelograms.*

Proof. The lemma follows from Proposition 3.11. The 3-valent vertices of C are dual to the triangles of Subdiv_f while the intersection of edges are dual to the parallelograms (see e.g. the right-hand side of Figure 3 and the corresponding lattice subdivision in Figure 7. □

We have the following formula which expresses the genus of a simple tropical curve V_f in terms of the number r of triangles in Subdiv_f.

Lemma 4.3. *If a curve $V_f \subset \mathbb{R}^2$ is simple then $g(V_f) = \frac{r-x}{2} + 1$.*

Proof. Let Δ_0 be the number of vertices of Subdiv_f while Δ_1 and Δ_2 be the number of its edges and (2-dimensional) polygons. Out of the Δ_2 2-dimensional polygons r are triangles and $(\Delta_2 - r)$ are parallelograms.

We have

$$\chi(V_f) = -2\Delta_2 + \Delta_1.$$

Note that $3r + 4(\Delta_2 - r) = 2\Delta_1 - x$. Thus, $\Delta_1 = \frac{3}{2}r + 2(\Delta_2 - r) + \frac{x}{2}$ and

$$g(V_f) = 1 - \chi(V_f) = 1 + \frac{r - x}{2}.$$

□

4.2. Tropical general positions of points in $\mathbb{R}^2$.

Definition 4.4. Points $p_1, \ldots, p_k \in \mathbb{R}^2$ are said to be *in general position tropically* if for any tropical curve $C \subset \mathbb{R}^2$ of genus g and with x ends such that $k \leq g + x - 1$ and $p_1, \ldots, p_k \in C$ we have the following conditions.

- The curve C is simple (see Definition 2.9).
- Points $p_1, \ldots, p_k$ are disjoint from the vertices of C.

Example 4.5. Two distinct points $p_1, p_2 \in \mathbb{R}^2$ are in general position tropically if and only if the slope of the line in $\mathbb{R}^2$ passing through p_1 and p_2 is irrational.

Remark 4.6. Note that we can always find a curve of genus $n + 1 - x$ passing through $p_1, \ldots, p_k$. For such a curve we can take a non-reducible curve consisting of k lines with rational slope each passing through its own point p_j. This curve has $2k$ ends while its genus is $1 - k$.

Proposition 4.7. *Any subset of a set of points in general position tropically is itself in general position tropically.*

Proof. Suppose the points $p_1, \ldots, p_j$ are not in general position. Then there is a curve C with x ends of genus $j + 2 - x$ passing through $p_1, \ldots, p_j$ or of genus $j + 1 - x$ but with a non-generic behavior with respect to $p_1, \ldots, p_j$. By Remark 4.6 there is a curve C' passing through $p_{j+1}, \ldots, p_k$ of genus $k - j + 1 - x'$. The curve $C \cup C'$ supplies a contradiction. □

Proposition 4.8. *For each $\Delta \subset \mathbb{R}^2$ the set of configurations $\mathcal{P} = \{p_1, \ldots, p_k\} \subset \mathbb{R}^2$ such that there exists a curve C of degree Δ such that the conditions of Definition 4.4 are violated by C is closed and nowhere dense.*

Proof. Note that we have only finitely many combinatorial types of tropical curves of genus g with the Newton polygon Δ. For each such combinatorial type we have an $(x+g-1)$-dimensional family of simple curves or a smaller-dimensional family otherwise. For a fixed C each of the $(x+g)$ points p_j can vary in a 1-dimensional family on C or in a 0-dimensional family if p_j is a vertex of C. Thus the dimension of the space of "bad" configurations $\mathcal{P} \in \mathrm{Sym}^k(\mathbb{R}^2)$ is at most $2k - 1$. □

Corollary 4.9. *The configurations $\mathcal{P} = \{p_1, \dots, p_k\}$ in general position tropically form a dense set which can be obtained as an intersection of countably many open dense sets in* $\mathrm{Sym}^k(\mathbb{R}^2)$.

4.3. **Tropical enumerative problem in $\mathbb{R}^2$.** To set up an enumerative problem we fix the *degree*, i.e. a polygon $\Delta \subset \mathbb{R}^2$ with $s = \#(\partial\Delta \cap \mathbb{Z}^2)$ and the *genus* $g \in \mathbb{Z}$. Consider a configuration $\mathcal{P} = \{p_1, \dots, p_{s+g-1}\} \subset \mathbb{R}^2$ of $s + g - 1$ points in tropical general position. Our goal is to count tropical curves $C \subset \mathbb{R}^2$ of genus g and degree Δ such that $C \supset \mathcal{P}$.

Proposition 4.10. *There exists only finitely many such C. Furthermore, each end of such C is of weight 1, so C has s ends.*

Finiteness follows from Lemma 4.20 proved in the next subsection. If C has ends whose weight is greater than 1 then the number of ends is smaller than s and the existence of C contradicts to the general position of $\mathcal{P}$. Recall that since $\mathcal{P}$ is in general position any such C is also simple and the vertices of C are disjoint from $\mathcal{P}$.

Example 4.11. Let $g = 0$ and Δ be the quadrilateral whose vertices are $(0,0)$, $(1,0)$, $(0,1)$ and $(2,2)$ (so that the number s of the lattice points on the perimeter $\partial\Delta$ is 4). For a configuration $\mathcal{P}$ of 3 points in $\mathbb{R}^2$ pictured in Figure 8 we have 3 tropical curves passing. In Figure 9 the corresponding number is 2.

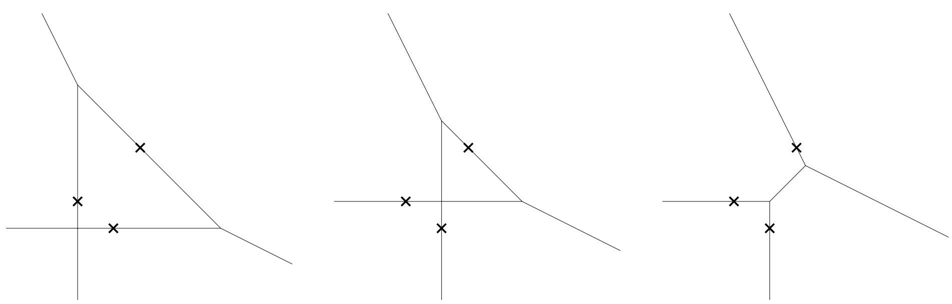

FIGURE 8. Tropical curves through a configuration of 3 points, $N^{\mathrm{irr}}_{\mathrm{trop}}(g, \Delta) = 5$.

Definition 4.12. The multiplicity $\mu(C)$ of a tropical curve $C \subset \mathbb{R}^2$ of degree Δ and genus g passing via $\mathcal{P}$ equals to the product of the multiplicities of all the 3-valent vertices of C (see Definition 2.16).

Definition 4.13. We define the number $N^{\mathrm{irr}}_{\mathrm{trop}}(g, \Delta)$ to be the number of irreducible tropical curves of genus g and degree Δ passing via $\mathcal{P}$

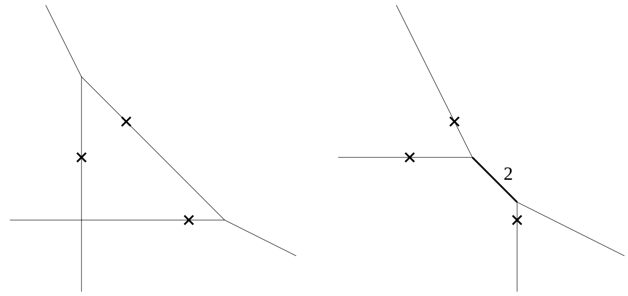

FIGURE 9. Tropical curves through another configuration of 3 points. Note that the bounded edge in the right-hand curve has weight 2, $N^{\mathrm{irr}}_{\mathrm{trop}}(g,\Delta)=5$.

where each such curve is counted with multiplicity μ from Definition 4.12. Similarly we define the number $N_{\mathrm{trop}}(g,\Delta)$ to be the number of all tropical curves of genus g and degree Δ passing via $\mathcal{P}$ where each such curve is counted with multiplicity μ from Definition 4.12.

The following theorem is a corollary of Theorem 1. In fact, no proof of it entirely within Tropical Geometry is known.

Theorem 4.14. *The numbers $N_{\mathrm{trop}}(g,\Delta)$ and $N^{\mathrm{irr}}_{\mathrm{trop}}(g,\Delta)$ are finite and do not depend on the choice of $\mathcal{P}$.*

E.g. the 3-point configurations from Figures 8 and 9 have the same number $N^{\mathrm{irr}}_{\mathrm{trop}}(g,\Delta)$.

4.4. **Forests in the polygon Δ.** Recall that every vertex of $C\subset\mathbb{R}^2$ corresponds to a polygon of the dual lattice subdivision of the Newton polygon Δ while every edge of C corresponds to an edge of the dual subdivision Subdiv_C (see Proposition 3.11). If $C\subset\mathbb{R}^2$ is a tropical curve passing through $\mathcal{P}$ then we can mark the k edges of Subdiv_C dual to $p_1,\dots,p_k$. Let $\Xi\subset\Delta$ be the union of the marked k edges.

Definition 4.15. The *combinatorial type* of a tropical curve C passing via the configuration $\mathcal{P}$ is the lattice subdivision Subdiv_C together with the graph $\Xi\subset\Delta$ formed by the k marked edges of this subdivision.

Proposition 4.16. *The graph $\Xi\subset\Delta$ is a forest (i.e. a disjoint union of trees).*

Proof. From the contrary, suppose that Ξ contains a cycle $Z\subset\Xi$ formed by q edges. Without loss of generality we may assume that $p_1,\dots,p_q$ are the marked points on the edges of C dual to Z. We claim that $p_1,\dots,p_q$ are *not* in tropical general position which leads to a contradiction with Proposition 4.7. To show this we exhibit a curve of non-positive genus with q tails at infinity passing through $p_1, \dots, p_q$.

Consider the union D of the (closed) polyhedra from Subdiv_C that are enclosed by Z. Let

$$C^D = \bigcup_{\Delta' \subset D} U^{\Delta'},$$

where $U^{\Delta'}$ is the stratum of C dual to Δ' (see Proposition 3.11). The set C^D can be extended to a tropical curve $\tilde{C}^D$ by extending all non-closed bounded edges of C^D to infinity. These extensions can intersect each other if D is not convex so the Newton polygon of $\tilde{C}^D$ is $\mathrm{ConvexHull}(D)$. In other words, $\tilde{C}^D$ is given by a tropical polynomial $\tilde{f}^D(x) = \sum_{j \in \mathrm{ConvexHull}(D)} b_j x^j$ with some choice of $b_j \in \mathbb{R}_{\mathrm{trop}}$ (note that $b_j = a_j$ if $j \in D$). Consider the polynomial

$$\tilde{f}^Z(x) = \sum_{j \in \partial(\mathrm{ConvexHull}(D))} b_j x^j.$$

We have $V_{\tilde{f}^Z} \ni p_1, \dots, p_q$ since $V_{\tilde{f}^D} \ni p_1, \dots, p_q$. On the other hand the genus of $V_{\tilde{f}^Z}$ is non-negative by Lemma 4.3 since no vertex of $\mathrm{Subdiv}_{\tilde{f}^D}$ is in the interior of D. □

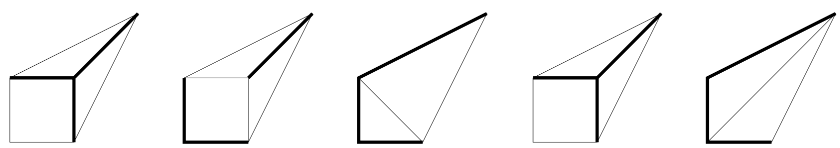

FIGURE 10. Forests corresponding to the curves passing through the marked points from Figures 8 and 9.

4.5. **The tropical curve minus the marked points.**

The following lemma strengthens the Proposition 4.16.

Lemma 4.17. *Let C be a simple curve of genus g and degree Δ. Suppose that C is parameterized by $h : \Gamma \to \mathbb{R}^2$.*

- *Suppose that C passes through a configuration $\mathcal{P}$ of $s + g - 1$ points in general position. Then each component K of $\Gamma \smallsetminus h^{-1}(\mathcal{P})$ is a tree and the closure of $h(K) \subset \mathbb{R}^2$ has exactly one end at infinity.*
- *Conversely, suppose that $\mathcal{P} \subset C$ is a finite set such that each component K of $\Gamma \smallsetminus h^{-1}(\mathcal{P})$ is a tree and the closure of $h(K) \subset \mathbb{R}^2$ has exactly one end at infinity. Then the combinatorial type*

of $(C, \mathcal{P})$ is realized by $(C, \mathcal{P}')$, where C' is a curve of genus g and degree Δ and $\mathcal{P}'$ is a configuration of $x + g - 1$ points in tropical general position. Here x is the number of ends of C.

Proof. Each component K of $C \setminus \mathcal{P}$ in the first part of the statement has to be a tree. Otherwise we can reduce the genus of C by the same trick as in Proposition 4.16 keeping the number of ends of C the same. This leads to a contradiction with the assumption that $\mathcal{P}$ is in general position. Also similarly to Proposition 4.16 we get a contradiction if $h(K)$ is bounded. If $h(K)$ has more than one end then by Theorem 2.10 it can be deformed keeping the marked points from $h(\bar{K}) \setminus h(K) \subset \mathcal{P}$ fixed. This supplies a contradiction with Proposition 4.10.

For the second part let us slightly deform $\mathcal{P}$ to bring it to a tropical general position. We can deform $h : \Gamma \to \mathbb{R}^2$ at each component K individually to ensure $C' \supset \mathcal{P}'$. □

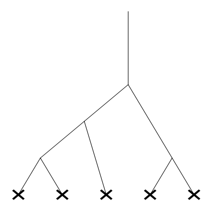

FIGURE 11. A sketch of a typical component of $\Gamma \setminus h^{-1}(\mathcal{P})$.

Lemma 4.17 allows to extend the forest Ξ from Proposition 4.16 to a tree $\mathcal{X} \subset \Delta$ that spans all the vertices of Subdiv_C in the case when the number of points in the configuration $\mathcal{P}$ is $s + g - 1$.

Each parallelogram Δ' corresponds to an intersection of two edges E and E' of Γ. Lemma 4.17 allows to orient the edges of each component of $\Gamma \setminus h^{-1}(\mathcal{P})$ consistently toward the end at infinity. Let $U' \subset \mathbb{R}^2$ be a small disk with a center at $E \cap E'$. Each component of $U' \setminus (E \cup E')$ corresponds to a vertex of Δ'. Two of these components are distinguished by the orientations of E and E'. One is adjacent to the sources of the edges while the other is adjacent to the sinks. We connect the corresponding edges of Δ' with an edge.

We form the graph $\mathcal{X} \subset \Delta$ by taking the union of Ξ with such edges for all the parallelograms from Subdiv_C.

Proposition 4.18. *The graph $\mathcal{X}$ is a tree that contains all the vertices of* Subdiv_C.

Proof. Suppose that Subdiv_C does not contain parallelograms. Then $\mathcal{X} = \Xi$. Let $K \subset \mathcal{X}$ be a component of $\mathcal{X}$. If there exists a vertex

$v \in \mathrm{Subdiv}_C$ not contained in K then we can form a 1-parametric family of curves of genus g and degree Δ passing via $\mathcal{P}$. Indeed, let $f(x) =$ "$\sum \beta_j x^j$ be the tropical polynomial that defines C. To get rid of the ambiguity in the choice of f we assume that j runs over only the vertices of Subdiv_C and that $\beta_{j_0} = 0$ for a choice of the base index $j_0 \in K$. Let us deform the coefficient β_v. If v belongs to component K' of Ξ different from K then we also inductively deform coefficients at the other vertices of Subdiv_C that belong to K' to make sure that the curve C' corresponding to the result of deformation still contains $\mathcal{P}$. Clearly the genus of C' is still g. Thus we get a contradiction to the assumption that $\mathcal{P}$ is in tropically general position.

We can reduce the general case to this special case by the following procedure. For each parallelogram $\Delta' \in \mathrm{Subdiv}_C$ consider a point p' that is obtained by a small shift of the intersection $E \cap E'$ to the source component of $U' \smallsetminus (E \cup E')$. Let $\mathcal{P}' \supset \mathcal{P}$ be the resulting configuration. The curve C can be deformed to a curve $C' \supset \mathcal{P}'$ with the corresponding forest Ξ' equal to $\mathcal{X}$. □

Corollary 4.19. *If $\mathcal{P}$ consists of $s + g - 1$ points then the forest Ξ from Proposition 4.16 contains all the vertices of* Subdiv_C.

4.6. **Uniqueness in a combinatorial type.** Enumeration of tropical curves is easier than that of complex or real curves thanks to the following lemma.

Lemma 4.20. *In each combinatorial type of marked tropical curves of genus g with x ends there is either one or no curves passing through $\mathcal{P} \in \mathbb{R}^2$ as long as $\mathcal{P}$ is a configuration of $x + g - 1$ points in general position.*

Proof. The tropical immersions to $\mathbb{R}^2$ of the graph Γ defined by the combinatorial type are described by linear equations (cf. the proof of Theorem 2.10). The condition that the image of a particular edge of Γ contains p_j is another linear condition. We have a zero-dimensional set satisfying to all conditions by Theorem 2.14.

Note that even if the combinatorial type is generic we still may have no curves passing through $\mathcal{P}$ since this linear system of equations defined not in the whole $\mathbb{R}^n$ but in an open polyhedral domain there by Lemma 3.14. □

5. Algebraic curves in $(\mathbb{C}^*)^2$

5.1. **Enumerative problem in $(\mathbb{C}^*)^2$.** As in subsection 4.3 we fix a number $g \in \mathbb{Z}$ and a convex lattice polygon $\Delta \subset \mathbb{R}^2$. As before let $s = \#(\partial\Delta \cap \mathbb{Z}^2)$. Let $\mathcal{P} = \{p_1, \dots, p_{s+g-1}\} \in (\mathbb{C}^*)^2$ be a configuration of points in general position. A complex algebraic curve $C \subset (\mathbb{C}^*)^2$ is defined by a polynomial $f : (\mathbb{C}^*)^2 \to \mathbb{C}$ with complex coefficients. As in the tropical set-up we refer to the Newton polygon Δ of f as the *degree of C*.

Definition 5.1. We define the number $N^{\mathrm{irr}}(g, \Delta)$ to be the number of irreducible complex curves of genus g and degree Δ passing via $\mathcal{P}$. Similarly we define the number $N(g, \Delta)$ to be the number of all complex curves of genus g and degree Δ passing via $\mathcal{P}$.

Note that here we count every relevant complex curve simply, i.e. with multiplicity 1.

Proposition 5.2. *For a generic choice of $\mathcal{P}$ the numbers $N^{\mathrm{irr}}(g, \Delta)$ and $N(g, \Delta)$ are finite and do not depend on $\mathcal{P}$.*

This proposition is well-known. Nevertheless, later in this section we reduce the invariance of $N^{\mathrm{irr}}(g, \Delta)$ and $N(g, \Delta)$ to the invariance of certain Gromov-Witten numbers.

In modern Mathematics there are two ways to interpret the numbers $N^{\mathrm{irr}}(g, \Delta)$ and $N(g, \Delta)$. A historically older interpretation is via the degree of Severi varieties. A more recent interpretation (introduced in [11]) is via the Gromov-Witten invariants. In both interpretations it is convenient to consider the compactification of the problem provided by the toric surface associated to the polygon Δ.

5.2. **Toric surfaces and Severi varieties.** Recall that a convex polygon Δ defines a compact toric surface $\mathbb{C}T_\Delta \supset (\mathbb{C}^*)^2$, see e.g. [6]. (Some readers may be more familiar with the definition of toric surfaces by fans, in our case the fan is formed by the dual cones at the vertices of Δ, see Figure 12.) The sides of the polygon Δ correspond to the divisors in $\mathbb{C}T_\Delta \smallsetminus (\mathbb{C}^*)^2$. These divisors intersect at the points corresponding to the vertices of Δ. This surface is non-singular if every vertex of Δ is *simple*, i.e. its neighborhood in Δ is mapped to a neighborhood of the origin in the positive quadrant angle under a composition of an element of $SL_2(\mathbb{Z})$ and a translation in $\mathbb{R}^2$. Non-simple vertices of Δ correspond to singularities of $\mathbb{C}T_\Delta$.

Example 5.3. Let Δ_d be the convex hull of $(d, 0)$, $(0, d)$ and $(0, 0)$. We have $\mathbb{C}T_{\Delta_d} = \mathbb{CP}^2$ no matter what is d. If $\Delta = [0, r] \times [0, s]$, $r, s \in \mathbb{N}$ then $\mathbb{C}T_\Delta = \mathbb{CP}^1 \times \mathbb{CP}^1$ no matter what are r and s. All the vertices of such polygons are simple.

If $\Delta^{\frac{3}{2}} = \mathrm{ConvexHull}\{(0, 0), (2, 1), (1, 2)\}$ then $\mathbb{C}T_\Delta = \mathbb{CP}^2/\mathbb{Z}^3$, where the generator of $\mathbb{Z}^3$ acts on $\mathbb{CP}^2$ by $[x : y : z] \mapsto [x : e^{\frac{2\pi i}{3}} y : e^{\frac{4\pi i}{3}} z]$.

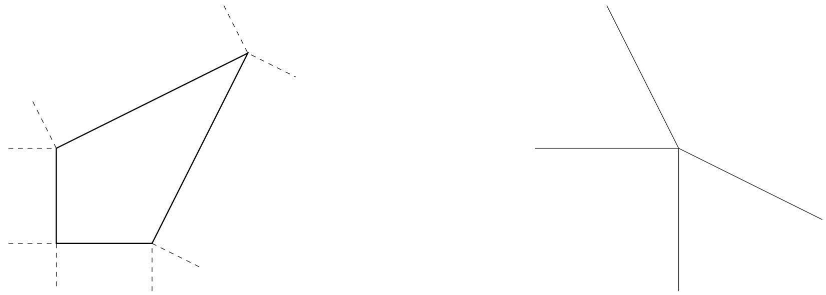

FIGURE 12. A polygon and its normal fan.

This action has 3 fixed points which give the singularities of $\mathbb{C}T_{\Delta^{\frac{3}{2}}}$ corresponding to the three (non-simple) vertices of Δ.

In addition to a complex structure (which depends only on the dual fan) the polygon Δ defines a holomorphic linear bundle $\mathcal{H}$ over $\mathbb{C}T_\Delta$. Let $\mathcal{L}_\Delta = \Gamma(\mathcal{H})$ be the vector space of sections of $\mathcal{H}$. The projective space $\mathbb{P}\mathcal{L}_\Delta$ is our system of the curves. Note that it can be also considered as the space of all holomorphic curves in $\mathbb{C}T_\Delta$ such that their homology class is Poincaré dual to $c_1(\mathcal{H})$.

Returning to Example 5.3 we note that Δ_d gives us the projective curves of degree d. The polygon $[0,r]\times[0,s]$ gives us the curves of bidegree (r,s) in the hyperboloid $\mathbb{CP}^1\times\mathbb{CP}^1$. The polygon $\Delta^{\frac{3}{2}}$ gives us the images in $\mathbb{C}T_{\Delta^{\frac{3}{2}}}$ of the cubic curves in $\mathbb{CP}^2$ that are invariant with respect to the $\mathbb{Z}_3$-action.

Any curve in $\mathcal{L}_\Delta$ is the closure in $\mathbb{C}T_\Delta$ of the zero sets of polynomials whose Newton polygon is contained in Δ. Thus $\dim\mathcal{L}_\Delta = \#(\Delta\cap\mathbb{Z}^2) = s+l$ (see (4)) and $\dim\mathbb{P}\mathcal{L}_\Delta = s+l-1$. A general curve $C\subset\mathbb{C}T_\Delta$ from $\mathbb{P}\mathcal{L}_\Delta$ is a smooth curve that is transverse to $\mathbb{C}T_\Delta\smallsetminus(\mathbb{C}^*)^2$ (this means that it is transverse to all divisors corresponding to the sides of $\partial\Delta$ and does not pass through their intersection points).

By the genus formula we have $g(C)=l=\#(\operatorname{Int}\Delta\cap\mathbb{Z}^2)$ for a smooth curve C in $\mathbb{P}\mathcal{L}_\Delta$. However singular curves have smaller geometric genus. More precisely let $C\subset\mathbb{C}T_\Delta$ be the curve from $\mathbb{P}\mathcal{L}_\Delta$ and let $\tilde{C}\to C$ be its normalization. We define *the geometric genus* as $g(C)=\frac{1}{2}(2-\chi(\tilde{C}))$. Note that if C is not irreducible then $\tilde{C}$ is disconnected and then $g(C)$ may take a negative value.

Fix a number $g\in\mathbb{Z}$. The curves of genus not greater than g form in the projective space $\mathbb{P}\mathcal{L}_\Delta$ an algebraic variety known as *the Severi variety* of $\mathbb{C}T_\Delta$. This variety may have several components. E.g. if $\mathbb{C}T_\Delta$ has an exceptional divisor E corresponding to a side $\Delta_E\subset\Delta$ then reducible curves $E\subset C'$ where C' corresponds to the polygon

$\Delta' = \mathrm{ConvexHull}((\Delta \smallsetminus \Delta_E) \cap \mathbb{Z}^2)$ form a component (or a union of components) of the Severi variety. Such components correspond to smaller polygons $\Delta' \subset \Delta$. We are interested only in those components that correspond to the polygon Δ itself.

Definition 5.4. *The irreducible Severi variety* $\Sigma_g^{\mathrm{irr}} \subset \mathbb{P}\mathcal{L}_\Delta$ corresponding to Δ of genus g is formed by all irreducible curves whose Newton polygon is Δ and genus is not more then g. *The Severi variety* $\Sigma_g \subset \mathbb{P}\mathcal{L}_\Delta$ corresponding to Δ of genus g is formed by all curves whose Newton polygon is Δ and genus is not more then g. Clearly, $\Sigma_g \supset \Sigma_g^{\mathrm{irr}}$.

Note that Σ_g is empty unless $g \leq l$. If $g = l$ we have $\Sigma_g = \mathbb{P}\mathcal{L}$. If $g = l - 1$ then Σ_g is the (generalized) Δ-discriminant variety. It is sometimes convenient to set $\delta = l - g$. Similarly to [3] it can be shown that Σ_g is the closure in $\mathbb{P}\mathcal{L}$ of immersed nodal curves with δ ordinary nodes. In the same way, Σ_g^{irr} is the closure in $\mathbb{P}\mathcal{L}$ of irreducible immersed nodal curves with δ ordinary nodes.

It follows from the Riemann-Roch formula that Σ_g and Σ_g^{irr} have pure dimension $s + g - 1$. The Severi numbers $N^{\mathrm{irr}}(g, \Delta)$ and $N(g, \Delta)$ can be interpreted as the degrees of Σ_g^{irr} and Σ_g in $\mathbb{P}\mathcal{L}$.

Example 5.5. Suppose $\Delta = \Delta_d$ so that $\mathbb{C}T_\Delta = \mathbb{CP}^2$. We have $s = 3d$ and $l = \frac{(d-1)(d-2)}{2}$. The number $N^{\mathrm{irr}}(g, \Delta)$ is the number of genus g, degree d (not necessarily irreducible) curves passing through $3d + g - 1$ generic points in $\mathbb{CP}^2$.

The formula $N(l - 1, \Delta_d) = 3(d - 1)^2$ is well-known as the degree of the discriminant (cf. [6]). (More generally, if $\mathbb{C}T_\Delta$ is smooth then $N(l-1, \Delta) = 6\,\mathrm{Area}(\Delta) - 2\,\mathrm{Length}(\partial\Delta) + \#\,\mathrm{Vert}\,\Delta$, where $\mathrm{Length}(\partial\Delta) = s$ is the "lattice length" of $\partial\Delta$ and $\#\,\mathrm{Vert}\,\Delta$ is the number of vertices, see [6]).

An elegant recursive formula for $N^{\mathrm{irr}}(0, \Delta_d)$ was found by Kontsevich [11]. Caporaso and Harris [3] found an algorithm for computing $N(g, \Delta_d)$ for arbitrary g. See [23] for computations for some other rational surfaces, in particular, the Hirzebruch surfaces (this corresponds to the case when Δ is a trapezoid).

5.3. **Gromov-Witten invariants.** Let Δ be a polygon with simple vertices so that $\mathbb{C}T_\Delta$ is a smooth 4-manifold. This manifolds is equipped with a symplectic form ω_Δ defined by Δ. The linear system $\mathcal{L}_\Delta$ gives an embedding $\mathbb{C}T_\Delta \subset \mathbb{P}\mathcal{L}_\Delta \approx \mathbb{CP}^m$. This embedding induces ω_Δ. As we have already seen Δ also defines a homology class $\beta_\Delta \in H_2(\mathbb{C}T_\Delta)$, it is the homology class of the curves from $\mathbb{P}\mathcal{L}_\Delta$.

To define the Gromov-Witten invariants of genus g one takes a generic almost-complex structure on $\mathbb{C}T_\Delta$ that is compatible with ω_Δ

and count the number of pseudo-holomorphic curves of genus g via generic $s+g-1$ points in $\mathbb{C}T_\Delta$ in the following sense (see [11] for a precise definition). [1] Consider the space $\mathcal{M}_{g,s+g-1}(\mathbb{C}T_\Delta)$ of all stable (i.e. those with finite automorphism group) parameterized pseudo-holomorphic curves with $s+g-1$ marked points. Evaluation at each marked point produces a map $\mathcal{M}_{g,s+g-1}(\mathbb{C}T_\Delta) \to \mathbb{C}T_\Delta$. With the help of this map we can pull back to $\mathcal{M}_{g,s+g-1}(\mathbb{C}T_\Delta)$ any cohomology class in $\mathbb{C}T_\Delta$, in particular the cohomology class of a point. Doing so for each of the $s+g-1$ point and taking the cup-product of the resulting classes we get the Gromov-Witten invariant $I^{\mathbb{C}T_\Delta}_{g,s+g-1,\beta_\delta} < pt^{\otimes^{s+g-1}} >$.

The result is invariant with respect to deformations of the almost-complex structure. In many cases it is useful to pass to a generic almost-complex structure to make sure that for any stable curve $C \to \mathbb{C}T_\Delta$ passing through our points we have $H^1(C, N_{C/\mathbb{C}T_\Delta}) = 0$. But we have this condition automatically if $\mathbb{C}T_\Delta$ is a smooth Fano surface (or, equivalently, all exceptional divisors have self-intersection -1), cf. e.g. [23] for details. (However, if $\mathbb{C}T_\Delta$ has exceptional divisors of self-intersection -2 and less we need either to perturb the almost-complex structure or to consider a virtual fundamental class.)

In particular, if $\mathbb{C}T_\Delta$ is $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$ (these are the only smooth toric surfaces without exceptional divisors, in other words minimal Fano) then the Gromov-Witten invariants coincide with the corresponding numbers $N^{\mathrm{irr}}(g,\Delta)$. The Gromov-Witten invariant of $\mathbb{CP}^2$ of degree d is equal to $N^{\mathrm{irr}}(g,\Delta_d)$. The Gromov-Witten invariant of $\mathbb{CP}^1 \times \mathbb{CP}^1$ of bidegree (r,s) is equal to $N^{\mathrm{irr}}(g,[0,r]\times[0,s])$.

The following proposition computes the Gromov-Witten invariants for non-minimal smooth Fano toric surfaces in terms of the numbers $N^{\Delta,\delta}$. Let $E_1,\dots,E_n$ be all the sides of Δ that correspond to the exceptional divisors of $\mathbb{C}T_\Delta$. For a subcollection $E_{j_1},\dots,E_{j_k}$ of such exceptional divisors we denote

$$\Delta_{j_1,\dots,j_k} = \mathrm{ConvexHull}((\Delta \smallsetminus \bigcup E_{j_k}) \cap \mathbb{Z}^2).$$

We have the following proposition.

Proposition 5.6. *If $\mathbb{C}T_\Delta$ is a smooth Fano surface then its Gromov-Witten invariant*

$$I^{\mathbb{C}T_\Delta}_{g,s+g-1,\beta_\delta} < pt^{\otimes^{s+g-1}} >= \sum_{k=0}^{n} \sum_{j_1<\dots<j_k} N^{\mathrm{irr}}(g,\Delta_{j_1,\dots,j_k}).$$

[1] These are the Gromov-Witten invariants evaluated on the cohomology classes dual to a point, this is the only non-trivial case for surfaces. In this discussion we completely ignore the gravitational descendants.

In particular, if $\mathbb{C}T_\Delta$ is a minimal smooth Fano surface (i.e. isomorphic to $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$) then

$$I^{\mathbb{C}T_\Delta}_{g,s+g-1,\beta_\delta} < pt^{\otimes s+g-1} >= N^{\mathrm{irr}}(g,\Delta).$$

Proof. Since $\mathbb{C}T_\Delta$ is Fano no component of $\Sigma_{l-\delta}$ has dimension higher then $s+g-1$. We have to add the degrees of all components consisting of curves of the type $E_{j_1} \cup \dots \cup E_{j_k} \cup C$, where the Newton polygon of C is $\Delta_{j_1,\dots,j_k}$. □

The Gromov-Witten invariants corresponding to disconnected curves (i.e. to the numbers $N(g,\Delta)$) are sometimes called multicomponent Gromov-Witten invariants.

6. Statement of the main theorems

6.1. **Enumeration of complex curves.** Recall that the numbers $N^{\mathrm{irr}}_{\mathrm{trop}}(g,\Delta)$ and $N_{\mathrm{trop}}(g,\Delta)$ were introduced in Definition 4.13 and a priori they depend on a choice of a configuration $\mathcal{R} \subset \mathbb{R}^2$ of $s+g-1$ points in general position.

Theorem 1. *For any generic choice $\mathcal{R}$ we have $N^{\mathrm{irr}}_{\mathrm{trop}}(g,\Delta) = N^{\mathrm{irr}}(g,\Delta)$ and $N_{\mathrm{trop}}(g,\Delta) = N(g,\Delta)$.*

Furthermore, there exists a configuration $\mathcal{P} \subset (\mathbb{C}^*)^2$ of $s+g-1$ points in general position such that for every tropical curve C of genus g and degree Δ passing through $\mathcal{R}$ we have $\mu(C)$ distinct complex curves of genus g and degree Δ passing through $\mathcal{P}$. These curves are distinct for distinct C and are irreducible if C is irreducible.

The following is an efficient way to compute $N_{\mathrm{trop}}(g,\Delta)$ (and therefore also $N(g,\Delta)$). Let us choose a configuration $\mathcal{P}$ to be contained in an affine line $\mathcal{A} \subset \mathbb{R}^2$. Furthermore, we make sure that the order of $\mathcal{P}$ coincides with the order on $\mathcal{A}$ and that $d(p_{j+1},p_j) >> d(p_j,p_{j-1})$. These conditions ensure that $\mathcal{P}$ is in tropically general position as long as the slope of $\mathcal{A}$ is irrational. Furthermore, these conditions specify the combinatorial types of the tropical curves of genus g and degree Δ passing via $\mathcal{P}$ (see Definition 4.15). We shall see that for this choice of $\mathcal{P}$ the forests corresponding to such combinatorial types are all paths connecting a pair of vertices of Δ. This observation can be used to compute $N(g,\Delta)$ once the irrational slope of $\mathcal{A}$ is chosen.

6.2. Counting of complex curves by lattice paths.

Definition 6.1. A path $\gamma : [0,n] \to \mathbb{R}^2$, $n \in \mathbb{N}$, is called a *lattice path* if $\gamma|_{[j-1,j]}$, $j = 1,\dots,n$ is an affine-linear map and $\gamma(j) \in \mathbb{Z}^2$, $j \in 0,\dots,n$.

Clearly, a lattice path is determined by its values at the integer points. Let us choose an auxiliary linear map

$$\lambda : \mathbb{R}^2 \to \mathbb{R}$$

that is irrational, i.e. such that $\lambda|_{\mathbb{Z}^2}$ is injective. (If an affine line $\mathcal{A} \subset \mathbb{R}^2$ is already chosen we take λ to be the orthogonal projection onto $\mathcal{A} \subset \mathbb{R}^2$.) Let $p, q \in \Delta$ be the vertices where $\alpha|_\Delta$ reaches its minimum and maximum respectively. A lattice path is called λ-*increasing* if $\lambda \circ \gamma$ is increasing.

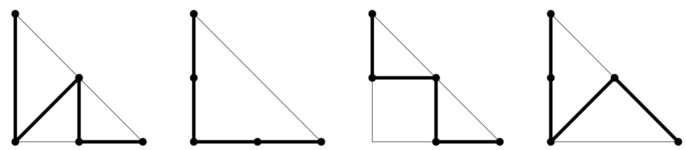

FIGURE 13. All λ-increasing paths for a triangle with vertices $(0,0)$, $(2,0)$ and $(0,2)$ where $\lambda(x,y) = x - \epsilon y$ for a small $\epsilon > 0$ and $\delta = 1$.

The points p and q divide the boundary $\partial\Delta$ into two increasing lattice paths

$$\alpha^+ : [0, n_+] \to \partial\Delta \ \text{ and } \ \alpha^- : [0, n_-] \to \partial\Delta.$$

We have $\alpha_+(0) = \alpha_-(0) = p$, $\alpha_+(n_+) = \alpha_-(n_-) = q$, $n_+ + n_- = m - l + 3$. To fix a convention we assume that α_+ goes clockwise around $\partial\Delta$ wile α_- goes counterclockwise.

Let $\gamma : [0, n] \to \Delta \subset \mathbb{R}^2$ be an increasing lattice path such that $\gamma(0) = p$ and $\gamma(n) = q$. The path γ divides Δ into two closed regions: Δ_+ enclosed by γ and α_+ and Δ_- enclosed by γ and α_-. Note that the interiors of Δ_+ and Δ_- do not have to be connected.

We define the positive (resp. negative) multiplicity $\mu_\pm(\gamma)$ of the path γ inductively. We set $\mu_\pm(\alpha_\pm) = 1$. If $\gamma \neq \alpha_\pm$ then we take $1 \le k \le n-1$ to be the smallest number such that $\gamma(k)$ is a vertex of $\Delta_\pm$ with the angle less than π (so that $\Delta_\pm$ is locally convex at $\gamma(k)$).

If such k does not exist we set $\mu_\pm(\gamma) = 0$. If k exist we consider two other increasing lattice paths connecting p and q $\gamma' : [0, n-1] \to \Delta$ and $\gamma'' : [0, n] \to \mathbb{R}^2$. We define γ' by $\gamma'(j) = \gamma(j)$ if $j < k$ and $\gamma'(j) = \gamma(j+1)$ if $j \ge k$. We define γ'' by $\gamma''(j) = \gamma(j)$ if $j \neq k$ and $\gamma''(k) = \gamma(k-1) + \gamma(k+1) - \gamma(k) \in \mathbb{Z}^2$. We set

$$\mu_\pm(\gamma) = 2\,\mathrm{Area}(T)\mu_\pm(\gamma') + \mu_\pm(\gamma''), \tag{5}$$

where T is the triangle with the vertices $\gamma(k-1)$, $\gamma(k)$ and $\gamma(k+1)$. The multiplicity is always integer since the area of a lattice triangle is half-integer.

Note that it may happen that $\gamma''(k) \notin \Delta$. In such case we use a convention $\mu_\pm(\gamma'') = 0$. We may assume that $\mu_\pm(\gamma')$ and $\mu_\pm(\gamma'')$ is already defined since the area of $\Delta_\pm$ is smaller for the new paths. Note that $\mu_\pm = 0$ if $n < n_\pm$ as the paths γ' and γ'' are not longer than γ.

We define *the multiplicity of the path* γ as the product $\mu_+(\gamma)\mu_-(\gamma)$. Note that the multiplicity of a path connecting two vertices of Δ does not depend on λ. We only need λ to determine whether a path is increasing.

Example 6.2. Consider the path $\gamma : [0, 8] \to \Delta_3$ depicted on the extreme left of Figure 14. This path is increasing with respect to $\lambda(x, y) = x - \epsilon y$, where $\epsilon > 0$ is very small.

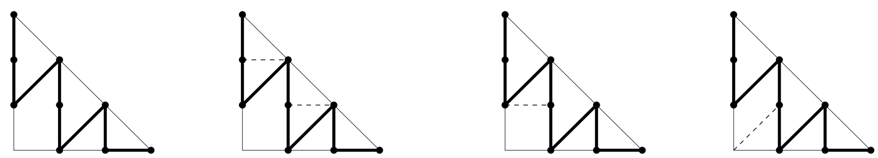

FIGURE 14. A path γ with $\mu_+(\gamma) = 1$ and $\mu_-(\gamma) = 2$.

Let us compute $\mu_+(\gamma)$. We have $k = 2$ as $\gamma(2) = (0, 1)$ is a locally convex vertex of Δ_+. We have $\gamma''(2) = (1, 3) \notin \Delta_3$ and thus $\mu_+(\gamma) = \mu_+(\gamma')$, since $\mathrm{Area}(T) = \frac{1}{2}$. Proceeding further we get $\mu_+(\gamma) = \mu_+(\gamma') = \cdots = \mu_+(\alpha_+) = 1$.

Let us compute $\mu_-(\gamma)$. We have $k = 3$ as $\gamma(3) = (1, 2)$ is a locally convex vertex of Δ_-. We have $\gamma''(3) = (0, 0)$ and $\mu_-(\gamma'') = 1$. To compute $\mu_-(\gamma') = 1$ we note that $\mu_-((\gamma')') = 0$ and $\mu_-((\gamma')'') = 1$. Thus the full multiplicity of γ is 2.

Recall that we fixed an (irrational) linear function $\lambda : \mathbb{R}^2 \to \mathbb{R}$ and this choice gives us a pair of extremal vertices $p, q \in \Delta$.

Theorem 2. *The number $N_{\mathrm{trop}}(g, \Delta)$ is equal to the number (counted with multiplicities) of λ-increasing lattice paths $[0, s + g - 1] \to \Delta$ connecting p and q.*

Furthermore, there exists a configuration $\mathcal{R} \in \mathbb{R}^2$ of $s+g-1$ points in tropical general position such that each λ-increasing lattice path encodes a number of tropical curves of genus g and degree Δ passing via $\mathcal{R}$ of total multiplicity $\mu(\gamma)$. These curves are distinct for distinct paths.

Example 6.3. Let us compute $N(0, \Delta) = 5$ for the polygon Δ depicted on Figure 15 in two different ways. Using $\lambda(x, y) = -x + \epsilon y$ for a small $\epsilon > 0$ we get the left two paths depicted on Figure 15. Using $\lambda(x, y) = x + \epsilon y$ we get the three right paths. The corresponding multiplicities are shown under the path. All other λ-increasing paths have zero multiplicity.

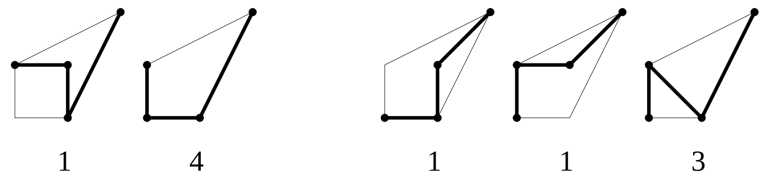

FIGURE 15. Computing $N(1, \Delta) = 5$ in two different ways.

In the next example we use $\lambda(x, y) = x - \epsilon y$ as the auxiliary linear function.

Example 6.4. Figure 16 shows a computation of the well-known number $N(1, \Delta_3)$. This is the number of rational cubic curves through 8 generic points in $\mathbb{CP}^2$.

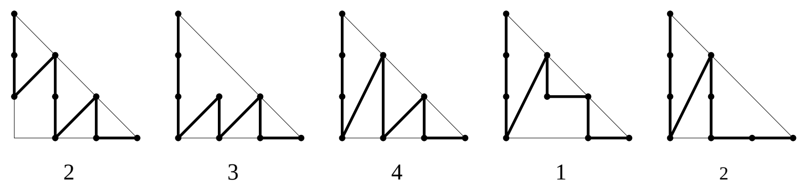

FIGURE 16. Computing $N(0, \Delta_3) = 12$.

Example 6.5. Figure 17 shows a computation of a less well-known number $N(1, \Delta_4)$. This is the number of genus 1 quartic curves through 12 generic points in $\mathbb{CP}^2$.

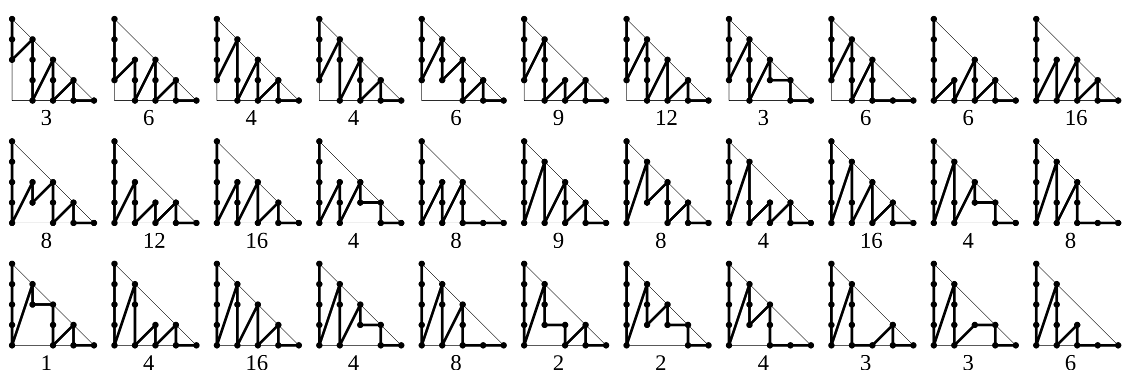

FIGURE 17. Computing $N(1, \Delta_4) = 225$.

6.3. **Enumeration of real curves.** Let $\mathcal{P} = \{p_1, \dots, p_{s+g-1}\} \subset (\mathbb{R}^*)^2$ be a configuration of points in general position. We have a total of $N(g, \Delta)$ complex curves of genus d and degree Δ in $(\mathbb{C}^*)^2$ passing through $\mathcal{P}$. Some of these curves are defined over $\mathbb{R}$ while others come in complex conjugated pairs.

Definition 6.6. We define the number $N^{\mathrm{irr}}_{\mathbb{R}}(g, \Delta, \mathcal{P})$ to be the number of irreducible real curves of genus g and degree Δ passing via $\mathcal{P}$. Similarly we define the number $N_{\mathbb{R}}(g, \Delta, \mathcal{P})$ to be the number of all real curves of genus g and degree Δ passing via $\mathcal{P}$.

Unlike the complex case the numbers $N^{\mathrm{irr}}_{\mathbb{R}}(g, \Delta, \mathcal{P})$ and $N_{\mathbb{R}}(g, \Delta, \mathcal{P})$ do depend on the choice of $\mathcal{P}$. We have

$$0 \leq N_{\mathbb{R}}(g, \Delta, \mathcal{P}) \leq N(g, \Delta)$$

while $N_{\mathbb{R}}(g, \Delta, \mathcal{P}) \equiv N(g, \Delta) \pmod 2$ and, similarly, $0 \leq N^{\mathrm{irr}}_{\mathbb{R}}(g, \Delta, \mathcal{P}) \leq N^{\mathrm{irr}}(g, \Delta)$ and $N^{\mathrm{irr}}_{\mathbb{R}}(g, \Delta, \mathcal{P}) \equiv N^{\mathrm{irr}}(g, \Delta) \pmod 2$.

Tropical geometry allows to compute $N_{\mathbb{R}}(g, \Delta, \mathcal{P})$ and $N_{\mathbb{R}}(g, \Delta, \mathcal{P})$ for some configurations $\mathcal{P}$. Note that one can extract a *sign* sequence $\{\mathrm{Sign}(p_j)\}_{j=1}^{s-g+1} \subset \mathbb{Z}_2^2$ from $\mathcal{P} = \{p_j\}_{j=1}^{s-g+1}$ by taking the coordinatewise sign. Some signs choices turn out to be equivalent.

Accordingly we can enhance the tropical configuration data by the adding a choice of signs that take values in certain equivalence classes in $\mathbb{Z}_2^2$. Equivalence is determined by the slope of the edge containing r_j. Suppose that r_j belongs to an edge of C whose weight is w_j and slope is $(x_j, y_j) \in \mathbb{Z}^2$ (by that we mean that (x_j, y_j) is a primitive integer vector parallel to the edge of r_j).

We define the set S_j as the quotient of $\mathbb{Z}_2^2$ by the equivalence relation $(X, Y) \sim (X + w_j x_j, Y + w_j y_j)$, $X, Y \in \mathbb{Z}_2$. Thus if the weight w_j is even then $S_j = \mathbb{Z}_2^2$ but if S_j is odd then S_j is a 2-element set.

Definition 6.7. A *signed* tropical configuration of points

$$\mathcal{R} = \{(r_1, \sigma_1), \dots, (r_{s+g-1}, \sigma_{s+g-1})\}$$

is a collection of $s - g + 1$ points r_j in the tropical plane $\mathbb{R}^2$ together with a choice of signs $\sigma_j \in S_j$.

We denote with $|\mathcal{R}| = \{r_1, \dots, r_{s+g-1}\} \subset \mathbb{R}^2$ the resulting configuration after forgetting the signs.

Our next goal is to define the *real multiplicity* of a tropical curve passing through a signed configuration. This multiplicity *does* depend on the choice of signs.

Let $h : \Gamma \to \mathbb{R}^2$ be a tropical curve of genus g and degree Δ passing through $|\mathcal{R}|$. Recall that by Lemma 4.17 each component of $\Gamma \smallsetminus h^{-1}(|\mathcal{R}|)$ is a tree with a single end at infinity (see Figure 11).

We define the real tropical multiplicity $\mu_{\mathbb{R}}(T, \mathcal{R})$ of each component T of $\Gamma \smallsetminus h^{-1}(|\mathcal{R}|)$ inductively. Let A and B be two 1-valent vertices of T corresponding to marked points r_a and r_b such that the edges adjacent to A and B meet at a 3-valent vertex C (see Figure 18).

FIGURE 18. Inductive reduction of the components of $\Gamma \smallsetminus h^{-1}(|\mathcal{R}|)$ in the definition of real multiplicity

Form a new tree T' by removing the edges $[A, C]$ and $[B, C]$ from T. The number of 1-valent vertices of T' is less by one (C becomes a new 1-valent vertex while A and B disappear). By the induction assumption the real multpilicity of T' is already defined for any choice of signs. All the finite 1-valent vertices of T' except for C have their signs induced from the signs of T. To completely equip T' with the signs we have to define the sign σ_d at the edge $[C, D]$.

Suppose that the slopes of $[A, C]$ and $[B, C]$ are (x_a, y_a) and (x_b, y_b) and their signs are σ_a and σ_b respectively. Suppose that $[C, D]$ is the third edge adjacent to C. Let S_a, S_b, S_d be the set of equivalence classes of signs corresponding to $[A, C]$, $[B, C]$ and $[C, D]$. Let w_a, w_b, w_d be their weights.

Definition 6.8. The sign at C and the real multiplicity of T are defined according to the following rules.

- Suppose that $w_a \equiv w_b \equiv 1 \pmod 2$ and $(x_a, y_a) \equiv (x_b, y_b) \pmod 2$. In this case we have $S_a = S_b$ so the signs σ_a and σ_b take values in the same set. Note that the weight of $[C, D]$ is even in this case. The sign r_d on such edge takes values in $S_d = \mathbb{Z}_2^2$. If $\sigma_a = \sigma_b$ then this sign can be presented by two distinct equivalent elements $\sigma_d^+, \sigma_d^- \in \mathbb{Z}_2^2$. Let T'_+ and T'_- be the trees equipped with the corresponding signs. We set

$$\mu_{\mathbb{R}}(T) = \mu_{\mathbb{R}}(T'_+) + \mu_{\mathbb{R}}(T'_-). \tag{6}$$

  If $\sigma_a \neq \sigma_b$ we set $\mu_{\mathbb{R}}(T) = 0$.
- Suppose that $w_a \equiv w_b \equiv 1 \pmod 2$ and $(x_a, y_a) \not\equiv (x_b, y_b) \pmod 2$. In this case the weight of $[C, D]$ is odd and the three sets S_a, S_b, S_d are all distinct. The sign $\sigma_d \in S_d$ is uniquely

determined by the condition that its equivalence class has common elements both with the equivalence class σ_a and with the equivalence class σ_b. Let T' be the tree equipped with this sign. We set

$$\mu_{\mathbb{R}}(T) = \mu_{\mathbb{R}}(T'). \tag{7}$$

- Suppose that one of the weights w_a and w_b is odd and the other is even. E.g. suppose that $w_a \equiv 1 \pmod 2$ and $w_b \equiv 0 \pmod 2$. In this case we have $w_d \equiv 1 \pmod 2$ and $S_a = S_d$ while $S_b = \mathbb{Z}_2^2$. If the equivalence class σ_a contains σ_b we set $\sigma_d = \sigma_a$ and

$$\mu_{\mathbb{R}}(T) = 2\mu_{\mathbb{R}}(T'), \tag{8}$$

where T' is equipped with the sign σ_d at $[C, D]$. If the equivalence class σ_a does not contain σ_b we set $\mu_{\mathbb{R}}(T) = 0$.

- Suppose that $w_a \equiv w_b \equiv 0 \pmod 2$. Then w_d is even and $S_a = S_b = S_d = \mathbb{Z}_2^2$. If $\sigma_a = \sigma_b$ we set $\sigma_d = \sigma_a$ and

$$\mu_{\mathbb{R}}(T) = 4\mu_{\mathbb{R}}(T'), \tag{9}$$

where T' is equipped with the sign σ_d at $[C, D]$. If $\sigma_a \neq \sigma_b$ we set $\mu_{\mathbb{R}}(T) = 0$.

Let $\mathcal{R}$ be a signed configuration of $s + g - 1$ points and $h : \Gamma \to \mathbb{R}^2$ be a tropical curve $C = h(\Gamma)$ of genus g and degree Δ passing via $|\mathcal{R}|$. The real multiplicity of a tropical curve passing is the product

$$\mu_{\mathbb{R}}(C, \mathcal{R}) = \prod_T \mu_{\mathbb{R}}(T),$$

where T runs over all the components of $\Gamma \smallsetminus h^{-1}(|\mathcal{R}|)$.

Proposition 6.9. *The real multiplicity $\mu_{\mathbb{R}}(C, \mathcal{R})$ is never greater than and has the same parity as the multiplicity of C from Definition 4.12.*

Proof. The proposition follows directly from Definition 6.8 by induction. The multiplicity of a three-valent vertex is odd if and only if all three adjacent edges have odd weights. This multiplicity is at least 2 if one of the adjacent edges has even weight. This multiplicity is at least 4 if all three adjacent edges have even weights. □

Similarly we have a real tropical enumerative number once the genus g, the degree Δ and a signed configuration $\mathcal{R}$ of $s + g - 1$ points is fixed.

Definition 6.10. We define the number $N^{\text{irr}}_{\text{trop},\mathbb{R}}(g, \Delta, \mathcal{R})$ to be the number of irreducible tropical curves of genus g and degree Δ passing via $|\mathcal{R}|$ counted with real multiplicities $\mu_{\mathbb{R}}(C, \mathcal{R})$. Similarly we define the number $N_{\text{trop},\mathbb{R}}(g,\Delta,\mathcal{R})$ to be the number of all tropical curves of genus g and degree Δ passing via $|\mathcal{R}|$ counted with real multiplicities $\mu_{\mathbb{R}}(C,\mathcal{R})$.

Theorem 3. *Suppose that $\mathcal{R}$ is a signed configuration of $s+g-1$ points in tropically general position. Then there exists a configuration $\mathcal{P} \subset (\mathbb{R}^*)^2$ of $s+g-1$ real points in general position such that $N^{\text{irr}}_{\mathbb{R}}(g,\Delta,\mathcal{P}) = N^{\text{irr}}_{\text{trop},\mathbb{R}}(g,\Delta,\mathcal{R})$ and $N_{\mathbb{R}}(g,\Delta,\mathcal{P}) = N_{\text{trop},\mathbb{R}}(g,\Delta,\mathcal{R})$.*

Furthermore, for every tropical curve C of genus g and degree Δ passing through $|\mathcal{R}|$ we have $\mu_{\mathbb{R}}(C,\mathcal{R})$ distinct real curves of genus g and degree Δ passing through $\mathcal{P}$. These curves are distinct for distinct C and are irreducible if C is irreducible.

Example 6.11. Let us choose the signs of $\mathcal{R}$ so that every σ_j contains $(+,+) \in \mathbb{Z}_2^2$ in its equivalence class. Let $g=0$ and Δ be the quadrilateral whose vertices are $(0,0)$, $(1,0)$, $(0,1)$ and $(2,2)$ as in Example 4.11. Then we have $N_{\text{trop},\mathbb{R}}(g,\Delta,\mathcal{R}) = 3$ for the configuration of 3 points from Figure 8 and $N_{\text{trop},\mathbb{R}}(g,\Delta,\mathcal{R}) = 5$ for the configuration of 3 points from Figure 9.

For other choices of signs of $\mathcal{R}$ we can get $N_{\text{trop},\mathbb{R}}(g,\Delta,\mathcal{R}) = 1$ for Figure 9 while $N_{\text{trop},\mathbb{R}}(g,\Delta,\mathcal{R}) = 3$ for Figure 8 for any sign choices.

6.4. **Counting of real curves by lattice paths.** Theorem 2 can be modified to give the relevant count of real curves. In order to do this we need to define the real multiplicity of a lattice path $\gamma : [0,n] \to \Delta$ connecting the vertices p and q once γ is equipped with signs.

Suppose $\gamma(j) - \gamma(j-1) = (y_j, x_j) \in \mathbb{Z}^2$, $j = 1,\dots,n$. Let $w_j \in \mathbb{N}$ be the GCD of y_j and x_j. Similarly to the previous subsection we define S_j to be the set obtained from $\mathbb{Z}_2^2$ by taking the quotient under the equivalence relation $(X,Y) \sim (X+x_j, Y+y_j)$, $X,Y \in \mathbb{Z}_2$. Let

$$\sigma = \{\sigma_j\}_{j=1}^n, \ \sigma_j \in S_j$$

be any choice of signs.

We set

$$\mu^{\mathbb{R}}_{\pm}(\gamma,\sigma) = a(T)\mu^{\mathbb{R}}_{\pm}(\gamma',\sigma') + \mu^{\mathbb{R}}_{\pm}(\gamma'',\sigma''). \tag{10}$$

The definition of the new paths γ', γ'' and the triangle T is the same as in subsection 13. The sign sequence for γ'' is $\sigma''_j = \sigma_j, j \neq k, k+1$, $\sigma''_k = \sigma_{k+1}$, $\sigma''_{k+1} = \sigma_k$. The sign sequence for γ' is $\sigma'_j = \sigma_j, j < k$, $\sigma'_j = \sigma_{j+1}, j > k$. We define the sign σ'_k and the function $a(T)$ (in a way similar to Definition 6.8) as follows.

- If all sides of T are odd we set $a(T) = 1$ and define the sign σ'_k (up to the equivalence) by the condition that the three equivalence classes of σ_k, σ_{k+1} and σ'_k do not share a common element.

- If all sides of T are even we set $a(T) = 0$ if $\sigma_{k-1} \neq \sigma_k$. In this case we can ignore γ' (and its sequence of signs). We set $a(T) = 4$ if $\sigma_k = \sigma_{k+1}$. In this case we define $\sigma'_k = \sigma_k = \sigma_{k+1}$.
- Otherwise we set $a(T) = 0$ if the equivalence classes of σ_k and σ_{k+1} do not have a common element. We set $a(T) = 2$ if they do. In the latter case we define the equivalence class of σ'_k by the condition that σ_k, σ_{k+1} and σ'_k have a common element. There is one exception to this rule. If the even side is $\gamma(k+1)-\gamma(k-1)$ then there are two choices for σ'_k satisfying the above condition. In this case we replace $a(T)\mu^{\mathbb{R}}_{\pm}(\gamma')$ in (10) by the sum of the two multiplicities of γ' equipped with the two allowable choices for σ'_k (note that this agrees with $a(T) = 2$ in this case).

Similar to subsection 13 we define $\mu^{\mathbb{R}}_{\pm}(\alpha_{\pm}) = 1$ and $\mu^{\mathbb{R}}(\gamma, \sigma) = \mu^{\mathbb{R}}_{+}(\gamma, \sigma)\mu^{\mathbb{R}}_{-}(\gamma, \sigma)$. As before $\lambda : \mathbb{R}^2 \to \mathbb{R}$ is a linear map injective on $\mathbb{Z}^2$ and p and q are the extrema of $\lambda|_{\Delta}$.

Theorem 4. *For any choice of $\sigma_j \in \mathbb{Z}_2^2$, $j = 1, \dots, s + g - 1$ there exists a configuration of $s + g - 1$ of generic points in the respective quadrants such that the number of real curves among the $N^{\Delta,\delta}$ relevant complex curves is equal to the number of λ-increasing lattice paths $\gamma : [0, s + g - 1] \to \Delta$ connecting p and q counted with multiplicities $\mu^{\mathbb{R}}$.*

Furthermore, each λ-increasing lattice path encodes a number of tropical curves of genus g and degree Δ passing via $\mathcal{R}$ of total real multiplicity $\mu^{\mathbb{R}}(\gamma, \sigma)$. These curves are distinct for distinct paths.

Example 6.12. Here we use the choice $\sigma_j = (+, +)$ so all the points z_j are in the positive quadrant $(\mathbb{R}_{>0})^2 \subset (\mathbb{R}^*)^2$. The first count of $N^{\Delta,1}$ from Example 6.3 gives a configuration of 3 real points with 5 real curves. The second count gives a configuration with 3 real curves as the real multiplicity of the last path is 1. Note also that the second path on Figure 15 changes its real multiplicity if we reverse its direction.

Example 6.4 gives a configuration of 9 generic points in $\mathbb{RP}^2$ with all 12 nodal cubics through them real. Example 6.5 gives a configuration of 12 generic points in $\mathbb{RP}^2$ with 217 out of the 225 quartics of genus 1 real. The path in the middle of Figure 17 has multiplicity 9 but real multiplicity 1. A similar computation shows that there exists a configuration of 11 generic points in $\mathbb{RP}^2$ such that 564 out of the 620 irreducible quartic through them are real.

6.5. **Different types of real nodes and the Welschinger invariant.** Let $V \subset (\mathbb{C}^*)^2$ be a curve defined over $\mathbb{R}$. In other words it is a curve invariant with respect to the involution of complex conjugation.

Suppose that V is nodal, i.e. all singularities of V are ordinary double points (nodes).

There are three types of nodes of V:

- Hyperbolic. These are the real nodes that locally are intersections of a pair of real branches. Such nodes are given by equation $z^2 - w^2 = 0$ for a choice of local real coordinates (z, w).
- Elliptic. These are the real nodes that locally are intersections of a pair of imaginary branches. Such nodes are given by equation $z^2 + w^2 = 0$ for a choice of local real coordinates (z, w).
- Imaginary. These are nodes at non-real points of V. Such nodes come in complex conjugate pairs.

This distinction was used in [25] in order to get a real curve counting *invariant* with respect to the initial configuration $\mathcal{P} \subset (\mathbb{R}^*)^2$. Indeed, let us modify the real enumerative problem from Definition 6.6 in the following way. Let $V \subset (\mathbb{C}^*)^2$ be a real nodal curve. Let $e(V)$ be the number of real elliptic nodes of V. We prescribe to V the sign equal to $(-1)^{e(V)}$. As usual we fix a genus g, a degree Δ and a configuration $\mathcal{P} \in (\mathbb{R}^*)^2$ of $s + g - 1$ points in general position.

Definition 6.13 (see [25])**.** We define the number $N^{\text{irr}}_{\mathbb{R},W}(g, \Delta, \mathcal{P})$ to be the number of irreducible real curves of genus g and degree Δ passing via $\mathcal{P}$ counted with signs. Similarly we define the number $N_{\mathbb{R},W}(g, \Delta, \mathcal{P})$ to be the number of all real curves of genus g and degree Δ passing via $\mathcal{P}$ counted with signs.

Theorem 5 (Welschinger [25])**.** *If $g = 0$ and $\mathbb{C}T_\Delta$ is smooth then the number $N^{\text{irr}}_{\mathbb{R},W}(g, \Delta, \mathcal{P})$ does not depend on the choice of $\mathcal{P}$.*

We call this number *the Welschinger invariant*. Theorem 3 can be modified to compute $N^{\text{irr}}_{\mathbb{R},W}(g, \Delta, \mathcal{P})$ and $N_{\mathbb{R},W}(g, \Delta, \mathcal{P})$.

Let $C \subset \mathbb{R}^2$ be a simple tropical curve. Recall that Definition 2.16 assigns a multiplicity $\text{mult}_V(C)$ to every 3-valent vertex $V \in C$.

Definition 6.14. We define

$$\text{mult}^{\mathbb{R},W}_V(C) = (-1)^{\frac{\text{mult}_V(C)-1}{2}}$$

if $\text{mult}_V(C)$ is odd and $\text{mult}^{\mathbb{R},W}_V(C) = 0$ if $\text{mult}_V(C)$ is even.

The *tropical Welschinger sign of C* is the product of $\text{mult}^{\mathbb{R},W}_V(C)$ over all 3-valent vertices of C.

As usual let us fix a genus g, a degree Δ and a configuration $\mathcal{R} \subset \mathbb{R}^2$ of $s+g-1$ points in tropically general position. Define $N^{\text{irr}}_{\text{trop},\mathbb{R}}(g, \Delta, \mathcal{R})$ to be the number of irreducible tropical curves of genus g and degree

Δ passing via $\mathcal{R}$ counted with the Welschinger sign. In a similar way define $N_{\mathrm{trop},\mathbb{R}}(g,\Delta,\mathcal{R})$ to be the number of all tropical curves of genus g and degree Δ passing via $\mathcal{R}$ counted with the Welschinger sign.

Theorem 6. *Suppose that $\mathcal{R} \subset \mathbb{R}^2$ is a configuration of $s+g-1$ points in tropically general position. Then there exists a configuration $\mathcal{P} \subset (\mathbb{R}^*)^2$ of $s+g-1$ real points in general position such that*

$$N_{\mathbb{R},W}(g,\Delta,\mathcal{P}) = N^{\mathrm{irr}}_{\mathrm{trop},\mathbb{R}}(g,\Delta,\mathcal{R})$$

and

$$N_{\mathbb{R},W}(g,\Delta,\mathcal{P}) = N_{\mathrm{trop},\mathbb{R}}(g,\Delta,\mathcal{R}).$$

Furthermore, for every tropical curve C of genus g and degree Δ passing through $\mathcal{R}$ we have a number of distinct real curves of genus g and degree Δ passing through $\mathcal{P}$ with the total sum of signs equal to $\nu_{\mathbb{R}}(C)$. These curves are distinct for distinct C and are irreducible if C is irreducible.

Example 6.15. Let $g=0$ and Δ be the quadrilateral with vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(2,2)$. In Figure 8 we have 3 real curves, two of them have the sign $+1$ and one has the sign -1. In Figure 9 we have 2 real curves, one of them has the sign $+1$ and one has the sign 0. In both cases we have $N_{\mathrm{trop},\mathbb{R}}(g,\Delta,\mathcal{R}) = 1$.

Theorem 2 can be adjusted to compute $N_{\mathrm{trop},\mathbb{R}}(g,\Delta,\mathcal{R}) = 1$ by lattice paths. Let $\gamma : [0,n] \to \Delta$ be a lattice path connecting the vertices p and q of Δ. Let us introduce the multiplicity $\nu^{\mathbb{R}}$ inductively, in a manner similar to our definition of the multiplicity $\mu(\gamma)$. Namely we set $\nu^{\mathbb{R}}(\gamma) = \nu^{\mathbb{R}}_{+}(\gamma)\nu^{\mathbb{R}}_{+}(\gamma)$. To define $\nu^{\mathbb{R}}_{\pm}(\gamma)$ we repeat the definition of $\mu_{\pm}(\gamma)$ but replace (5) with

$$\nu^{\mathbb{R}}_{\pm}(\gamma) = b(T)\nu^{\mathbb{R}}_{\pm}(\gamma') + \nu^{\mathbb{R}}_{\pm}(\gamma'').$$

Here we define $b(T) = 0$ if at least one side of T is even and $b(T) = (-1)^{\#(\mathrm{Int}\, T \cap \mathbb{Z}^2)}$ otherwise. The paths γ', γ'' and the triangle T are the same as in the inductive definition of $\mu_{\pm}$.

Theorem 7. *For any choice of an irrational linear map $\lambda : \mathbb{R}^2 \to \mathbb{R}$ there exists a configuration $\mathcal{P}$ of $s+g-1$ of generic points in $(\mathbb{R}^*)^2$ such that the number of real curves of genus g and degree Δ passing through $\mathcal{P}$ counted with the tropical Welschinger sign is equal to the number of λ-increasing lattice paths $\gamma : [0, s+g-1] \to \Delta$ connecting p and q counted with multiplicities $\nu^{\mathbb{R}}$.*

Furthermore, there exists a configuration $\mathcal{R} \in \mathbb{R}^2$ of $s+g-1$ points in tropical general position such that each λ-increasing lattice path encodes a number of tropical curves of genus g and degree Δ passing via $\mathcal{R}$ with *the sum of signs equal to $\nu^{\mathbb{R}}(\gamma)$. These curves are distinct for distinct paths.*

Example 6.16. In example 6.3 we have $N_{\mathbb{R},W}(\Delta,\mathcal{P}) = 1$ for some choice of $\mathcal{P}$. In example 6.4 we have $N^{\mathrm{irr}}_{\mathbb{R},W}(0,\Delta_3,\mathcal{P}) = 8$ for some choice of $\mathcal{P}$. In example 6.5 we have $N^{\mathrm{irr}}_{\mathbb{R},W}(1,\Delta_4,\mathcal{P}) = 93$ for some choice of $\mathcal{P}$. Note that by Theorem 5 the number $N^{\mathrm{irr}}_{\mathbb{R},W}(0,\Delta_3,\mathcal{P}) = 8$ does not depend on the choice of $\mathcal{P}$.

The following observation is due to Itenberg, Kharlamov and Shustin [8]. If $\lambda(x,y) = x - \epsilon y$ and $\Delta = \Delta_d$ of $\Delta = [0,d_1]\times[0,d_2]$ then $\nu^{\mathbb{R}}(\gamma) \geq 0$ for any λ-increasing path γ and, furthermore, any tropical curve encoded by γ by Theorem 7 has a non-negative tropical Welschinger sign. It is easy to show that there exist λ-increasing paths that encode irreducible tropical curves of non-zero tropical Welschinger signs. We get the following corollary for any $d, d_1, d_2 \in \mathbb{N}$.

Corollary 8. *For any generic configuration $\mathcal{P} \subset \mathbb{RP}^2$ of $3d-1$ points there exists an irreducible rational curve $V \subset \mathbb{RP}^2$ of degree d passing through $\mathcal{P}$.*

For any generic configuration $\mathcal{P} \subset \mathbb{RP}^1 \times \mathbb{RP}^1$ of $2d_1+2d_2-1$ points there exists an irreducible rational curve $V \subset \mathbb{RP}^1\times\mathbb{RP}^1$ of bidegree (d_1,d_2) passing through $\mathcal{P}$.

With the help of Theorem 7 Itenberg, Kharlamov and Shustin in [8] have obtained a non-trivial lower bound for the number of such rational curves. In particular, they have shown that for any generic configuration $\mathcal{P}$ of $3d-1$ points in $\mathbb{RP}^2$ there exists at least $\frac{d!}{2}$ rational curves of degree d passing via $\mathcal{P}$.

7. Connection between classical and tropical geometries

7.1. Degeneration of complex structure on $(\mathbb{C}^*)^2$.

Let $t > 1$ be a real number. We have the following self-diffeomorphism $(\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$

$$H_t : (z,w) \mapsto (|z|^{\frac{1}{\log t}}\frac{z}{|z|}, |w|^{\frac{1}{\log t}}\frac{w}{|w|}). \tag{11}$$

For each t this map induces a new complex structure on $(\mathbb{C}^*)^2$.

Here is a description of the complex structure induced by H_t in logarithmic polar coordinates $(\mathbb{C}^*)^2 \approx \mathbb{R}^2 \times iT^2$. (This identification is induced by the holomorphic logarithm $\mathcal{L}og$ from the identification $\mathbb{C}^2 \approx \mathbb{R}^2 \times i\mathbb{R}^2$.) If v is a vector tangent to iT^2 we set $J_t v = \frac{1}{\log(t)} iv$. Note that $J_t v$ is tangent to $\mathbb{R}^2$.

Clearly, a curve V_t is holomorphic with respect to J_t if and only if $V_t = H_t(V)$, where V is a holomorphic curve with respect to the standard complex structure, i.e. J_e-holomorphic. Let $\mathrm{Log} : (\mathbb{C}^*)^2 \to \mathbb{R}^2$ be the map defined by $\mathrm{Log}(z,w) = (\log|z|, \log|w|)$. We have

$$\mathrm{Log} \circ H_t = \mathrm{Log}_t .$$

Note that H_t corresponds to a $\log(t)$-contraction $(x,y) \mapsto (\frac{x}{\log(t)}, \frac{y}{\log(t)})$ under Log.

7.2. **Complex tropical curves in $(\mathbb{C}^*)^2$.** There is no limit (at least in the usual sense) for the complex structures J_t, $t \to \infty$. Nevertheless, as in section 6.3 of [16] we can define the J_∞-holomorphic curves which happen to be the limits of J_t-holomorphic curves, $t \to \infty$.

There are several way to define them. An algebraic definition is the shortest and involves varieties over a non-Archimedian field. Let K be the field of the (real-power) Puiseux series $a = \sum\limits_{j \in I_a} a_j t^j$, where $I_a \subset \mathbb{R}$ is a bounded from below set contained in a finite union of arithmetic progression. The field K is algebraically closed and of characteristic 0. The field K has a non-Archimedian valuation $\mathrm{val}(a) = -\min I_a$, $\mathrm{val}(a+b) \le \mathrm{val}(a) + \mathrm{val}(b)$.

As usual, we set $K^* = K \smallsetminus \{0\}$. The multiplicative homomorphism $\mathrm{val} : K^* \to \mathbb{R}$ can be "complexified" to $w : K^* \to \mathbb{C}^* \approx \mathbb{R} \times S^1$ by setting $w(a) = e^{\mathrm{val}(a) + i\arg(a_{\mathrm{val}(a)})}$. Applying this map coordinatewise we get the map $W : (K^*)^2 \to \mathbb{R}^2$. The image of an algebraic curve V_K under W turns out to be a J_∞-holomorphic curve (cf. Theorem 7.1).

Note the following special case. Let

$$V_K = \{z \in (\mathbb{C}^*)^2 \mid \sum_{j \in \mathrm{Vert}\,\Delta} a_j z^j\},$$

where $z \in (K^*)^2$, j runs over all vertices of Δ and a_j is such that $\mathrm{val}(a_j) = 0$. By Kapranov's theorem [9] $\mathrm{Val}(V_K) = \mathrm{Log}(W(V_K))$ is the tropical curve C_f defined by $f(x) =$ "$\sum\limits_{j \in \mathrm{Vert}\,\Delta} x^j$", $x \in \mathbb{R}^2$. Thus C_f is a union of rays starting from the origin and orthogonal to the sides of Δ, in other words it is the 1-skeleton of the normal fan to Δ. However, $W(V_K) \subset (\mathbb{C}^*)^2$ depends on the argument of the leading term of $a_j \in K^*$. Thus different choices of $W(V_K)$ give different *phases* for the lifts of C_f. We may translate C_f in $\mathbb{R}^2$ so that it has a vertex in a point $x \in \mathbb{R}^2$ instead of the origin. Corresponding translations of $W(V_K)$ give a set of possible lifts.

This allows to give a more geometric description of J_∞-curves. They are certain 2-dimensional objects in $(\mathbb{C}^*)^2$ which project to tropical curves under Log. Namely, let $C \subset \mathbb{R}^2$ be a tropical curve, $x \in C$ be a point and $U \ni x \in \mathbb{R}^2$ be a convex neighborhood such that $U \cap C$ is a cone over x (i.e. for every $y \in C \cap U$ we have $[x, y] \subset C$). Note that if x is a point on an open edge then it is dual (in the sense of 3.4) to a segment in Δ. If x is a vertex of C then it is dual to a 2-dimensional polygon in Δ. In both cases we denote the dual polygon with Δ'. We say that a 2-dimensional polyhedron $V_\infty \subset (\mathbb{C}^*)^2$ is $(C \cap U)$-*compatible* if $\mathrm{Log}^{-1}(U) \cap V_\infty = \mathrm{Log}^{-1}(U) \cap W$, where W is a translation of $W(V_K)$ while $V_K = \{z \in (\mathbb{C}^*)^2 \mid \sum_j a_j z^j\}$, j runs over some lattice points of Δ' and $\mathrm{val}(a_j) = 0$.

Theorem 7.1. *Let $V_\infty \subset (\mathbb{C}^*)^2$ be a polyhedral subcomplex. The following conditions are equivalent.*

- *$V_\infty = W(V_K)$, where $V_K \subset (K^*)^2$ is an algebraic curve.*
- *$C = \mathrm{Log}(V_\infty)$ is a tropical curve such that for every $x \in C$ there exists a small open convex neighborhood $x \in U \subset \mathbb{R}^2$ such that $\mathrm{Log}^{-1}(U) \cap V_\infty$ is $(C \cap U)$-compatible.*
- *V_∞ is the limit when $k \to \infty$ in the Hausdorff metric of a sequence of J_{t_k}-holomorphic curves V_{t_k} with $\lim_{k\to\infty} t_k = \infty$.*

Definition 7.2. Curves satisfying to any of the equivalent conditions of Theorem 7.1 are called *complex tropical curves*.

Theorem 7.1 allows to think of complex tropical curves both as tropical curves equipped with a *phase*, i.e. a lifting to $(\mathbb{C}^*)^2$, and as J_∞-holomorphic curves, i.e. as limits of J_t-holomorphic curves when $t \to \infty$.

Proof of Theorem 7.1. $1 \implies 2$. Let $f : (K^*)^2 \to K$ be the polynomial defining V_K. The image $C = \mathrm{Log}(V_\infty)$ is a tropical curve by Kapranov's Theorem [9]. Let $x \in C \subset (K^*)^2$. The lowest t-powers of $f(x)$ come only from the polygon $\Delta' \subset \Delta$ dual to the stratum containing x. The compatibility curve is given by the sum of the Δ'-monomials of f.

$2 \implies 1$. Consider the subdivision of Δ dual to the tropical curve C. The compatibility condition gives us a choice of monomials for each polygon $\Delta' \subset \Delta$ in the subdivision dual to C. However, the choice is not unique, due to the higher t-power contributions. On the other hand, a monomial corresponds to a lattice point of Δ which may belong to several subpolygons in the subdivision.

We have to choose the coefficients for the monomials so that they would work for all subpolygons of the subdivision. Let $j \in \Delta \cap \mathbb{Z}^2$. The coefficient a_j is a Puiseux series in t. The number $\mathrm{val}(a_j)$ is defined by the equation for the tropical curve C. We set $a_j = \alpha_j t^{\mathrm{val}(a_j)}$, where $\alpha_j \in \mathbb{C}^*$, i.e. our coefficient Puiseux series are actually monomials.

Namely, let Δ' be a polygon in the subdivision of Δ dual to C. A point x on the corresponding stratum of C is compatible with $W(\{f^{\Delta'} = 0\})$ for a polynomial $f^{\Delta'}$ over K with the Newton polygon Δ'. The curve $W(\{f^{\Delta'} = 0\})$ coincides with the curve $W(\{f^{\Delta'}_{\min} = 0\})$, where the polynomial $f^{\Delta'}_{\min}$ is obtained by replacing each coefficient series $a_j = a_j(t)$ of $f^{\Delta'}$ with its lower t-power monomial $\alpha_j t^{\mathrm{val}(a_j)}$. The polynomial $f^{\Delta'}_{\min}$ is well-defined up to a multiplication by a constant. For a proper choice of constants for different Δ' the coefficients given by different $f^{\Delta'}_{\min}$ for the same $j \in \Delta \cap \mathbb{Z}^2$ agree.

$1 \implies 3$. This implication is a version of the so-called *Viro patchworking* [24] in real algebraic geometry. If $f : (K^*)^2 \to K$ is the polynomial defining V_K then we can construct the sequence $V_{t_k} \subset (\mathbb{C}^*)^2$ in the following way, see [17]. Let $f^{t_k} : (\mathbb{C}^*)^2 \to \mathbb{C}$ be the complex polynomial obtained from f by plugging $t = t_k$ to the Puiseux series coefficients of f. We set V_k to be the image of the zero set of f^{t_k} by the self-diffeomorphism H_{t_k} defined in (11).

$3 \implies 1$. Propositions 3.9 and 8.2 imply that $\mathrm{Log}(V_\infty)$ is a tropical curve. Let $\alpha_j \in \mathbb{R}_{\mathrm{trop}}$ be the coefficients of the tropical polynomial defining $\mathrm{Log}(V_\infty)$. To find a presentaion $V_\infty = W(V_K)$ we take the polynomial with coefficients $\beta_j t^{\alpha_j} \in K$, $\beta_j \in S^1$.

To find β_j we note that if V_k is a J_{t_k}-holomorphic curve then $H^{-1}_{t_k}(V_k)$ is (honestly) holomorphic and is given by a complex polynomial $f_k = \sum\limits_j a^{V_k}_j z^j$. To get rid of the ambiguity resulting from multiplication by a constant we may assume that $a^{V_k}_{j'} = 1$ for a given $j' \in \Delta \cap \mathbb{Z}^2$ and for all sufficiently large k. We take $\beta_j = \lim\limits_{k\to\infty} \mathrm{Arg}(a^{V_k}_j)$. □

Let $\mathrm{Arg} : (\mathbb{C}^*)^2 \to S^1 \times S^1$ be defined by $\mathrm{Arg}(z_1, z_2) = (\arg(z_1), \arg(z_2))$.

Proposition 7.3. *Suppose that $V_\infty \subset (\mathbb{C}^*)^2$ is a complex tropical curve, $C = \mathrm{Log}(V_\infty) \subset \mathbb{R}^2$ is the corresponding "absolute value" tropical curve and $x \in C$ is either a vertex dual to a polygon $\Delta' \subset \Delta$ or a point on an open edge dual to an edge $\Delta' \subset \Delta$. We have $\mathrm{Arg}(\mathrm{Log}^{-1}(x) \cap V_\infty) = \mathrm{Arg}(V')$ for some holomorphic curve $V' \subset (\mathbb{C}^*)^2$ with the Newton polygon Δ'.*

This proposition follows from the second characterization of J_∞-holomorphic curves in Theorem 7.1.

7.3. **Parameterized complex tropical curves and their genus.**

Definition 7.4. A map $\psi : \tilde{V}_\infty \to V_\infty$ is called a *parameterized complex tropical curve* if

- $\tilde{V}_\infty$ is homeomorphic to a surface;
- for any $x \in C = \mathrm{Log}(V_\infty)$ the map $\psi|_{(\mathrm{Log}\circ\psi)^{-1}(x)}$:
  $(\mathrm{Log}\circ\psi)^{-1}(x) \to V_\infty \cap \mathrm{Log}^{-1}(x) \subset \mathrm{Log}^{-1}(x) \approx S^1 \times S^1$
  is conjugate to the map $\mathrm{Arg} : V' \to S^1 \times S^1$ from Proposition 7.3;
- for any $x \in C = \mathrm{Log}(V_\infty)$ there is a small convex neighborhood $x \in U \subset \mathbb{R}^2$ with the following property. For every component V' of $(\mathrm{Log}\circ\psi)^{-1}(U)$ there exists a complex tropical curve $V'' \subset (\mathbb{C}^*)^2$ such that $\psi(V') = V'' \cap \mathrm{Log}^{-1}(U)$.

Definition 7.5. The *genus* $g(V_\infty)$ of a complex tropical curve V_∞ is the minimal genus of $\tilde{V}_\infty$ among all parameterizations $\tilde{V}_\infty \to V_\infty$. As usual, for the case when $\tilde{V}_\infty$ is disconnected we use a convention

$$g(\tilde{V}_\infty) = 1 + \sum_{V'} (g(V') - 1),$$

where V' runs over all connected components of $\tilde{V}_\infty$.

Proposition 7.6. *If $V_\infty \subset (\mathbb{C}^*)^2$ is a complex tropical curve and $C = \mathrm{Log}(V_\infty) \subset \mathbb{R}^2$ is the corresponding tropical curve then $g(C) \le g(V_\infty)$.*

Proof. Any parameterized complex tropical curve $\tilde{V}_\infty \to V_\infty$ defines a parameterized tropical curve $h : \Gamma \to C$ with $g(\tilde{V}_\infty) \ge g(C)$. □

Remark 7.7. It can happen that $g(C) < g(V_\infty)$. For example take V^a_∞ to be the limiting curve for the family $1 + x + y + at^{-1}xy$, where $a \in \mathbb{C}$, $|a| = 1$, and $t > 0$, $t \to \infty$. The image $C = \mathrm{Log}(V^a_\infty)$ does not depend on the choice of a. We have $g(C) = -1$. If $a = 1$ then $g(V^a_\infty) = -1$, otherwise $g(V^a_\infty) = 0$.

Proposition 7.8. *For any open edge $E \subset C$ the inverse image $(\mathrm{Log}\circ\psi)^{-1}(E)$ is a collection $\tilde{E}$ of disjoint annuli. The image $\psi(\tilde{E})$ is a collection of disjoint open holomorphic annuli in $(\mathbb{C}^*)^2$. The restriction of $\psi_{\tilde{E}}$ to each component of the image is a covering map. The total degree of these coverings equals to the weight of E.*

Proof. Denote with Δ' the (1-dimensional) polygon dual to E in the subdivision of Δ associated to C. By definition V_∞ is Δ'-compatible. Any holomorphic curve of degree Δ' is a collection of disjoint annuli in $(\mathbb{C}^*)^2$. □

Definition 7.9. Let $\psi : \tilde{V}_\infty \to V_\infty$ be a parameterized complex tropical curve. Let E be an open edge of the tropical curve $\mathrm{Log}(V_\infty)$. The *elementary cylinder* A of V_∞ is a component of $(\mathrm{Log}\circ\psi)^{-1}(E)$. Its *weight* is the degree of the covering $\psi_A : A \to \psi(A)$.

Definition 7.10. A parameterized complex tropical curve $\psi : \tilde{V}_\infty \to V^\circ_\infty \subset (\mathbb{C}^*)^2$ parameterized by is called *simple* if $C = \operatorname{Log}(V_\infty)$ is a simple tropical curve, $g(V_\infty) = g(C)$ and for any open edge $E \subset C$ the inverse image $(\operatorname{Log} \circ \psi)^{-1}(E)$ is connected.

Remark 7.11. Recall that we denote with Δ_1 a triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 0)$. A complex tropical curve of degree Δ_1 is called a complex tropical *projective line* (cf. Definition 2.6). Note that any such curve is simple. Any two such curves can be identified by a (multiplicative) translation in $(\mathbb{C}^*)^2$. Furthermore, if $\mathcal{P} \subset (\mathbb{C}^*)^2$ is a configuration of 2 points such that $\operatorname{Log}(\mathcal{P}) \subset \mathbb{R}^2$ is tropically generic then there exists a unique complex tropical projective line passing through $\mathcal{P}$.

Let $C \supset \operatorname{Log}(\mathcal{P})$ be a tropical curve in $\mathbb{R}^2$ of genus g and degree Δ. Let μ be the multiplicity of C (see Definition 4.12) and let $x \leq s$ be the number of ends of C. Let $\mathcal{P} \subset (\mathbb{C}^*)^2$ be a configuration of $x + g - 1$ points such that $\operatorname{Log}(\mathcal{P}) \subset \mathbb{R}^2$ is in tropically general position.

Let V_∞ be a simple complex tropical curve passing via $\mathcal{P}$. Since $qqq = \operatorname{Log}(\mathcal{P})$ is in tropically general position each $p_j \in \mathcal{P}$ sits on an elementary cylinder. We prescribe the multiplicity to V_∞ equal to the product of the weights of all its elementary cylinders that contain points from $\mathcal{P}$. Then we have the following proposition.

Proposition 7.12. *There are μ simple complex tropical curves in $(\mathbb{C}^*)^2$ of genus g and degree Δ such that they project to C and pass via $\mathcal{P}$.*

Let Δ be a triangle and $C \subset \mathbb{R}^2$ be a tropical curve of degree Δ with no bounded edges. Such curve has genus 0 and 3 unbounded edges. (Note that if $\#(\partial\Delta \cap \mathbb{Z}^2) > 3$ then some of the unbounded edges have weight greater than 1.) Let $V\infty \subset (\mathbb{C}^*)^2$ be a simple complex tropical curve V_∞ projecting to C.

Lemma 7.13. *The curve $V_\infty \subset (\mathbb{C}^*)^2$ lifts to a complex tropical projective line under some linear map $M_\Delta : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$ that is self-covering of degree* $2\operatorname{Area}(\Delta)$.

Clearly, we have $\deg(M_\Delta) = 2\operatorname{Area}(\Delta)$ of such lifts.

Proof of Lemma 7.13. Consider an affine-linear surjection $\Delta_1 \to \Delta$. Let

$$L_\Delta : \mathbb{R}^2 \to \mathbb{R}^2$$

be the linear part of this map. Note that $\det(L_\Delta) = 2\operatorname{Area}(\Delta)$. The matrix of L_Δ written multiplicatively defines a map

$$M_\Delta : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2. \tag{12}$$

Alternatively we can define M_Δ as the map covered by $L_\Delta \otimes \mathbb{C} : \mathbb{C}^2 \to \mathbb{C}^2$ under $\exp : \mathbb{C}^2 \to (\mathbb{C}^*)^2$. We have $\deg(M_\Delta) = \det(L_\Delta) = 2\,\mathrm{Area}(\Delta)$.

Note that M_Δ extends to a holomorphic map

$$\bar{M}_\Delta : \mathbb{CP}^2 \to \mathbb{C}T_\Delta. \tag{13}$$

If $\Delta' \subset \partial\Delta$ is a side of Δ then $(\bar{M}_\Delta)^{-1}(\Delta')$ consists of $2\,\mathrm{Area}(\Delta)/l(\Delta')$ components, where $l(\Delta') = \#(\Delta' \cap \mathbb{Z}^2) - 1$ is the integer length of Δ'.

Since V_∞ is simple we have $\tilde{V_\infty}$ homeomorphic to sphere punctured in 3 points. Each puncture corresponds to a side of Δ. A loop around the puncture in V_∞ wraps around $\mathbb{C}T_{\Delta'}$ $l(\Delta')$ times. Therefore, V_∞ lifts under M_Δ. □

Proof of Theorem 7.12. Note that C is simple since $\mathrm{Log}(\mathcal{P})$ is in tropically general position.

We start with a special case. Suppose (as in Lemma 7.13) that $g = 0$, Δ is a triangle and C has 3 ends. Then $\mathcal{P}$ consists of two points.

By Lemma 7.13 any simple complex tropical curve over C is an image of a complex tropical projective line under M_Δ. The set $(M_\Delta)^{-1}(\mathcal{P})$ consists of $2\deg(M_\Delta) = 4\,\mathrm{Area}(\Delta)$ points. Each pair projecting to different points under M_Δ can be connected with a complex tropical projective line. We have a total of $(\deg(M_\Delta))^2$ of such lines and each of them projects to a simple complex tropical curve of degree Δ under M_Δ. Conversely, each such simple complex tropical curve lifts to d distinct lines. Therefore, we have $\deg(M_\Delta) = 2\,\mathrm{Area}(\Delta)$ distinct simple complex tropical curves projecting to C and passing via $\mathcal{P}$.

Now we are ready to treat general case. By Lemma 4.17 each component K of $\Gamma \smallsetminus h^{-1}(\mathcal{P})$ is a tree which contains one end at infinity. Here $h : \Gamma \to C$ is the parametrization of C. Let $A, B \in \mathcal{P}$ be two points that connect to the same 3-valent vertex in K as in Figure 18. Let $\Delta' \subset \Delta$ be the subpolygon dual to this 3-valent vertex of C. By our special case we have $2\,\mathrm{Area}(\Delta')$-compatible special case and can proceed inductively. □

7.4. **Polynomials to define complex tropical curves.** Let $V_K \subset (K^*)^2$ be a curve given by a non-Archimedean polynomial

$$f_K(z) = \sum_j \alpha_j z^j,$$

$z \in K^2$, $j \in \Delta$, $\alpha_j \in K^*$. Let $V_\infty = W(V_K) \subset (\mathbb{C}^*)^2$ be the corresponding complex tropical curve. Denote with f the tropical polynomial defining the projection $\mathrm{Log}(V) \subset \mathbb{R}^2$. Note that by Kapranov's theorem [9] $f(x) =$ "$c \sum\limits_j \mathrm{val}(\alpha_j) x^j$".

Proposition 7.14. *If j and j' are vertices of* Subdiv_f *then the ratio $\frac{w(\alpha_j)}{w(\alpha_{j'})}$ depends only on V_∞ and does not depend on the choice of V_K such that $W(V_K) = V_\infty$.*

Proof. Since Δ is connected we need to prove the proposition only in the case when j and j' are connected with an edge $\Delta' \in \mathrm{Subdiv}_f$. The complex tropical curve V_∞ is Δ'-compatible, therefore the proposition follows from the special case when $\Delta = \Delta'$ is 1-dimensional.

After an automorphism in $(\mathbb{C}^*)^2$ we may assume that $\Delta = \Delta' = [0, l]$, $l \in \mathbb{N}$. Then V_∞ is a collection of the elementary cylinders each defined by equation $z = z_k$ and of weight w_k. We have $\frac{w(\alpha_j)}{w(\alpha_{j'})} = \prod z_k^{w_k}$. □

If $j \mapsto -\mathrm{val}(\alpha_j)$ is strictly convex then to recover a complex tropical curve $w(V_K)$ it sufficed to know only $a_j = w(\alpha_j)$.

Proposition 7.15. *Let $f(z) = \sum\limits_{j \in \Delta} a_j z^j$, be a formal sum of monomials, $a_j \in \mathbb{C}^*$, where j runs over some lattice points of Δ. Suppose that $j \mapsto -\log|a_j|$ is strictly convex. Then f defines a complex tropical curve $V_\infty \subset (\mathbb{C}^*)^2$ of degree Δ. This correspondence satisfies to the following properties.*

- *Two different such polynomials define different complex tropical curves.*
- *Complex tropical curves defined in this way form an open and dense set in the space of all complex tropical curves of degree Δ.*

Proof. Form a polynomial

$$f^K(x, y) = \sum_{(j,k) \in \Delta \cap \mathbb{Z}^2} a_{jk} t^{\log|a_{jk}|} x^j y^k$$

defined over K. Define $V_\infty = w(\{(x, y) \in (K^*)^2 \mid f^K(x, y) = 0\}$. To see that different polynomials define different curves it suffices to note that for the corresponding tropical polynomial

$$f_{\mathrm{trop}}(x) = \text{“}\sum_{j \in \Delta} \log|a_j| x^j\text{”}$$

the subdivision $\mathrm{Subdiv}_{f_{\mathrm{trop}}}$ contains all lattice points of Δ as its vertices. In particular, no lattice point of Δ is contained in the interior of an edge of $\mathrm{Subdiv}_{f_{\mathrm{trop}}}$. For different polynomials we have different Δ'-compatible elements for some edge $\Delta' \in \mathrm{Subdiv}_{f_{\mathrm{trop}}}$.

On the other hand, any complex tropical curve can be obtained from a non-Archimedean curve $V_K \subset (K^*)^2$ given by a polynomial $f_K(z) = \sum\limits_j \alpha_j z^j$ such that the function $j \mapsto -\mathrm{val}(\alpha_j)$ is convex. The polynomials with strictly convex functions $j \mapsto -\mathrm{val}(\alpha_j)$ are open and dense among all such polynomials. □

Remark 7.16. If $V_K \subset (K^*)^2$ is given by a polynomial $f_K(z) = \sum\limits_j \alpha_j z^j$ such that the function $j \mapsto -\mathrm{val}(\alpha_j)$ is not strictly convex then the collection of complex numbers $a_j = w(\alpha_j)$ does not necessarily determine the complex tropical curve $w(V_K)$.

E.g. the polynomials $f_K(z, w) = z^2+3z+2$ and $f'_K(z, w) = z^2+z+1$ (treated as polynomials over K) produce the same collection of three complex numbers, namely $a_{(0,0)} = a_{(1,0)} = a_{(2,0)} = 1$. However we have $w(V_K) = \{(z, w) \in (\mathbb{C}^*)^2 \mid z = -1\}$ and $w(V'_K) = \{(z, w) \in (\mathbb{C}^*)^2 \mid z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\}$.

8. Counting holomorphic curves by tropical curves

8.1. **Complex amoebas in $\mathbb{R}^2$ and the key lemma.** Let $\mathcal{P} = \{p_1, \dots, p_n\} \subset (\mathbb{C}^*)^2$ be generic points such that the points

$$r_1 = \mathrm{Log}(p_1), \dots, r_n = \mathrm{Log}(p_n)$$

are in general position tropically. Denote $\mathcal{R} = \{r_1, \dots, r_n\} \subset \mathbb{R}^2$. Recall that we fix a Newton polygon $\Delta \subset \mathbb{R}^2$ and a genus $g \in \mathbb{N}$. There are $N(g, \Delta)$ holomorphic curves passing through $\mathcal{P}$ as long as $n = s + g - 1$. By Proposition 4.10 there are finitely many tropical curves

$$C_1, \dots, C_m \subset \mathbb{R}^2 \tag{14}$$

of genus g with the Newton polygon Δ and passing through $\mathcal{R}$. Note that m depends on the choice of the points x_j (unlike the invariant number $N_{\mathrm{trop}}(g, \Delta) \geq m$).

Proposition 8.1. *For generic t we have $N(g, \Delta)$ J_t-holomorphic curves passing through $\mathcal{P}$.*

Proof. Indeed, this number equals to the number of holomorphic (i.e. J_e-holomorphic) curves through the points $H_t(p_1), \dots, H_t(p_n)$. (These points are in general position for generic t since they are for $t = 1$.) □

Let $\mathcal{N}_\epsilon(C_j)$ be the ϵ-neighborhood of C_j for $\epsilon > 0$. Recall (see [6]) that the *amoeba* of a curve $V \subset (\mathbb{C}^*)^2$ is its image

$$\mathcal{A} = \mathrm{Log}(V) \subset \mathbb{R}^2.$$

Note that if V is a J_t-holomorphic curve then $\mathrm{Log}(V)$ can be obtained from the amoeba of some holomorphic curve by the $\log(t)$-contraction. This allows us to speak of the Newton polygons of J_t-holomorphic curves.

Proposition 8.2. *If V is a J_t-holomorphic curve whose Newton polygon is Δ then its amoeba $\mathcal{A} = \mathrm{Log}(V)$ contains a tropical curve C with the same Newton polygon Δ.*

Proof. If $t = 1$, i.e. C is holomorphic with respect to the standard holomorphic structure, then the statement follows from the theorem of Passare and Rullgård [18]. Recall (see [18]) that that if V is a complex curve defined by a polynomial $f(z,w) = \sum\limits_{j,k} a_{j,k} z^j w^k$ then the *spine* of its amoeba $\mathcal{A}$ is a tropical curve defined by a tropical polynomial $N_f^\infty(x,y)$ "$\sum\limits_{j,k} b_{j,k} x^j y^k$", where

$$b_{j,k} = \frac{1}{(2\pi i)^2} \int\limits_{\mathrm{Log}^{-1}(r)} \log|f(z,w)| \frac{dz}{z}\frac{dw}{w}$$

and $r \in \mathbb{R}^2 \smallsetminus \mathcal{A}$ is any point such that its *index* is (j,k). If there are no points in $\mathbb{R}^2 \smallsetminus \mathcal{A}$ of index (j,k) then the monomial $x^j y^k$ is omitted from N_f^∞.

It is shown in [18] that the tropical hypersurface defined by N_f^∞ is contained in $\mathcal{A}$. Clearly, the Newton polygon of N_f^∞ is Δ. To finish the proof we note that the image of a tropical curve under a homothety is tropical with the same Newton polygon. □

By Proposition 8.1 Theorem 1 follows from the following lemma.

Lemma 8.3. *There exist a sufficiently small $\epsilon > 0$ and a sufficiently large $t > 0$ with the following properties.*

- **a:** *If V is a J_t-holomorphic curve of genus g and degree Δ passing through $\mathcal{P}$ then its amoeba $\mathrm{Log}(V)$ is contained in $\mathcal{N}_\epsilon(C_j)$ for some $j = 1, \dots, n$.*
- **b:** *The multiplicity of each C_j from (14) (see Definition 4.12) is equal to the number of the J_t-holomorphic curves V of genus g and degree Δ passing through $\mathcal{P}$ and such that $\mathrm{Log}(V)$ is contained in $\mathcal{N}_\epsilon(C_j)$.*

8.2. **Proof of Lemma 8.3.a.** A holomorphic curve $V \subset (\mathbb{C}^*)^2$ is given by a polynomial

$$f(z,w) = \sum_{(j,k)\in\Delta} a_{j,k} z^j w^k.$$

To a curve $V \subset (\mathbb{C}^*)^2$ we may associate its *tropicalization* $V^{\text{trop}} \subset \mathbb{R}^2$ given by the tropical polynomial

$$f^{\text{trop}}(x,y) = \max_{(j,k)\in\Delta} jx + ky + \log|a_{j,k}|.$$

Proposition 8.6 and the following lemma imply Lemma 8.3.a.

Lemma 8.4. *The amoeba* $\operatorname{Log}(V)$ *is contained in the* ϵ*-neighborhood of* V^{trop}*, where* $\epsilon = \log(\#(\Delta \cap \mathbb{Z}^2))$ *is the number of monomials of* f*.*

Proof. Suppose that (x_0, y_0) is not contained in the ϵ-neighborhood of V^{trop}. Then there exists (j,k) such that

$$jx_0 + ky_0 + \log|a_{j,k}| > j'x_0 + k'y_0 + \log|a_{j',k'}| + \epsilon \tag{15}$$

for any $(j,k) \neq (j_{\max}, k_{\max})$. Indeed, the distance from (x,y) to the line $jx+ky+\log|a_{j,k}| = j'x+k'y+\log|a_{j',k'}|$ is greater than ϵ and the norm of the gradient of the function $(j-j')x+(k-k')y$ is at least 1 (as j, j', k, k' are all integers). Let (z,w) be a point such that $\operatorname{Log}(z,w) = (x_0,y_0)$. Exponentiating the inequality (15) we get $|a_{j',k'} z^{j'} w^{k'}| < \frac{1}{m}|a_{j,k} z^j w^k|$. By the triangle inequality $f(z,w) \neq 0$. □

Corollary 8.5. *The amoeba* $\operatorname{Log}(V_t)$ *of a* J_t*-holomorphic curve* $V_t = H_t(V)$ *is contained in the* ϵ*-neighborhood of* V^{trop}*, where* $\epsilon = \log_t(\#(\Delta \cap \mathbb{Z}^2))$*.*

The corollary is obtained by applying the $\log t$-contraction to Lemma 8.4.

Let $V_k \subset (\mathbb{C}^*)^2$, $k \in \mathbb{N}$, be a sequence of curves passing through $\mathcal{P}$ and such that V_k is a J_{t_k}-holomorphic curve for some $t_k > 0$, where $t_k \to \infty$, $k \to \infty$. As in the previous subsection we assume that the holomorphic curve $H_{t_k}^{-1}(V_k)$ is of genus g and has the Newton polygon Δ for each k. Denote with $\mathcal{A}_k = \operatorname{Log}(V_k)$ the amoeba of V_k.

Lemma 8.3.a follows from Corollary 8.5 and the following proposition.

Proposition 8.6. *There is a subsequence* V_{k_α}*,* $\alpha \in \mathbb{N}$*, such that the sets* $\mathcal{A}_{k_\alpha} \subset \mathbb{R}^2$ *converge in the Hausdorff metric in* $\mathbb{R}^2$ *to some tropical curve* C_j *from* (14)*.*

Proof. Consider the tropicalizations V_k^{trop}. By Proposition 3.9 we can extract a subsequence which converges to a tropical curve C. To prove the proposition it suffices to show that the Newton polyhedron of C is a tropical curve passing through $\mathcal{R}$ of genus g whose Newton polygon is Δ. Proposition 3.9 and Corollary 8.5 ensure convergence in the Hausdorff metric in $\mathbb{R}^2$.

We have $C \supset \mathcal{R}$ since $V_k \supset \mathcal{P}$ and thus $\mathcal{A}_k \supset \mathcal{R}$. The degree of C is a subpolygon $\Delta' \subset \Delta$ since C is the limit of curves of degree Δ. We want to prove that $\Delta' = \Delta$.

Choose a disk $D_R \subset \mathbb{R}^2$ of radius R so large that D_R contains all vertices of C. Furthermore, making R larger if needed we may assume that the extension of the exterior edges of $C \cap D_R$ beyond D_R do not intersect.

The Newton polygon of V_k^{trop} is Δ. Therefore, it has s ends. By Proposition 3.9 $V_k^{\text{trop}} \cap D_R$ approximates $C \cap D_R$. Therefore, for a large k $V_k^{\text{trop}} \supset \mathcal{R}'$ where $\mathcal{R}'$ is a configuration of $s+g-1$ points in tropically general position obtained by a small deformation of $\mathcal{R}$. We have $\Delta' = \Delta$ if and only if $V_k^{\text{trop}} \smallsetminus D_R$ is a disjoint union of rays (each going to ∞). If not, $C \smallsetminus D_R$ has a bounded edge connecting a point of $V_k^{\text{trop}} \cap \partial D_R$ with a vertex of V_k^{trop} outside of D_R. A change of the length of this edge produces a deformation of V_k^{trop} such that all curves in the family pass via $\mathcal{R}'$. This contradicts to the tropical general position of $\mathcal{R}'$.

Note that the genus of C cannot be smaller than g, otherwise the configuration $\mathcal{R}$ is not in general position. The genus of curves V_k^{trop} may be larger than g even though the genus of V_k is g. However, the genus of their limit C cannot be larger than g by Proposition 7.6. Therefore, the genus of C is g. Thus C has to be one of C_k from (14). $\square$

8.3. **Proof of Lemma 8.3.b.** Let $C \subset \mathbb{R}^2$ be one of the tropical curves C_j from (14) and μ be its multiplicity from Definition 4.12. Denote with Subdiv_C the lattice subdivision of Δ dual to C. Let

$$f_{\text{trop}} = \sum_{j \in \Delta \cap \mathbb{Z}^2} \beta_j x^j \tag{16}$$

be the tropical polynomial that defines C.

By Theorem 7.12 there are μ complex tropical curves projecting to C and passing via $\mathcal{P}$. Let V be one of them. Let us fix a "reference vertex" $j_0 \in \Delta$ so that the complex coefficient at the corresponding monomial will always be 1. We may assume that $\beta_{j_0} = 0$.

By Propositions 7.14 we have a well-defined coefficients $a_j \in \mathbb{C}$ for the vertices j of Subdiv_C. Note that since $C \supset \mathcal{R}$ and $\mathcal{R}$ is in tropically general position the number of ends of C is s and therefore

$$\partial\Delta \cap \mathbb{Z}^2 \subset \mathrm{Vert}(\mathrm{Subdiv}_C).$$

The coefficients a_j correspond to the complex tropical curve V. Here $a_{j_0} = 1$ and $-\log(a_j) = \beta_j$, where j runs over all vertices of Subdiv_C is the tropical polynomial that defines C.

Proposition 8.7. *The coefficients a_j do not depend on the choice of V. Thus these coefficients depend only on C and $\mathcal{P}$.*

Proof. We repeat the induction steps from the proof of Proposition 7.12. By Lemma 4.17 each component K of $\Gamma \smallsetminus h^{-1}(\mathcal{P})$ is a tree which contains one end at infinity.

Let $A, B \in \mathcal{P}$ be two points that connect to the same 3-valent vertex in K as in Figure 18. Then the points A, B are contained in the edges of C dual to the edges $[j, j']$, $[j', j'']$ such that the triangle with the vertices j, j', j'' belongs to Subdiv_C. By Propositions 7.14 we may recover $\frac{a_j}{a_{j'}}$ and $\frac{a_{j'}}{a_{j'}}$. Therefore, we know the ratio $\frac{a_j}{a_{j''}}$ and we can proceed further by induction. □

Consider the polynomial

$$f_t(z) = \sum_{j \in \mathrm{Subdiv}_C} \arg(a_j) t^{-\log|a_j|}.$$

The sum is taken over all vertices j of Subdiv_C. Define $V_t \subset (\mathbb{C}^*)^2$ by

$$V_t = H_t(\{z \in (\mathbb{C}^*)^2 \mid f_t(z) = 0\}.$$

For each $t > 1$ the curve V_t is J_t-holomorphic. For large values of t we have $\mathrm{Log}(V_t) \subset \mathcal{N}_\epsilon(C)$.

For large values of t we may consider the curve V_t as an *approximate* solution to the problem of finding the J_t holomorphic curves of genus g and degree Δ via $\mathcal{P}$. Indeed, most likely V_t is smooth (and therefore of genus $\#(\Delta \cap \mathbb{Z}^2) \geq g$) and does not contain $\mathcal{P}$. However it is close to a (singular) curve of genus g and is very close to the configuration $\mathcal{P}$. We need to find a genuine solution near this approximate one for large values $t >> 1$.

Recall that the amoebas of the (J_t-holomorphic) curves we search have to be contained in a small neighborhood of C while the curves themselves have to contain $\mathcal{P}$. For large $t >> 1$ this implies that such curve can be presented in the form $V_t = H_t(\{f_t^\zeta(z) = 0)$, where

$$f_t^\zeta = \sum_{j \in \Delta \cap \mathbb{Z}^2} \arg(\zeta_j) t^{-\log|\zeta_j|} z^j$$

and $\zeta \in \mathbb{C}^n$, $n = \#(\Delta \cap \mathbb{Z}^2) - 1$, are such that $|\zeta_j - a_j| < \epsilon'_j$ for $j \in \mathrm{Vert}(\mathrm{Subdiv}_C)$ while $-\log|\zeta_j| - \beta_j < \epsilon_j$ for $j \notin \mathrm{Vert}(\mathrm{Subdiv}_C)$. Here $\epsilon'_j > 0$ is some collection of small numbers. All ζ that comply to these conditions form a polydisk $\mathcal{D} \subset \mathbb{C}^n$.

Using the subdivision Subdiv_C we can essentially localize the problem. We need to find the number of such choices for ζ that $V_t^\zeta \supset \mathcal{P}$ and V_t has genus g. Let $\Delta' \subset \mathrm{Subdiv}_V$ be a 2-dimensional subpolygon whose dual vertex is $v' \in C$. Then we denote with $U'(\Delta') \subset \mathbb{R}^2$ a small open neighborhood of v'. Let $\Delta' \subset \mathrm{Subdiv}_V$ be an edge whose dual edge is $E' \in C$. In this case we denote with $U'(\Delta') \subset \mathbb{R}^2$ a small open neighborhood of E'. We assume that $\epsilon > 0$ in Lemma 8.3 is so small that

$$\mathcal{N}_\epsilon \subset \bigcup_{\Delta' \in \mathrm{Subdiv}_C} U'(\Delta').$$

Proposition 8.8. *Suppose that $V_t^\zeta \supset \mathcal{P}$ is a curve of genus g and t is large.*

- *If Δ' is a triangle then $V_t^\zeta \cap \mathrm{Log}^{-1}(U'(\Delta'))$ is a rational curve with 3 ends.*
- *If Δ' is a parallelogram then $V_t^\zeta \cap \mathrm{Log}^{-1}(U'(\Delta'))$ splits to two irreducible components. Each of the components is an embedded annulus.*
- *If Δ' is an edge then $V_t^\zeta \cap \mathrm{Log}^{-1}(U'(\Delta'))$ is an irreducible component that is an immersed annulus.*

Conversely, suppose that t is large, $\zeta \in \mathcal{D}$ and V_t^ζ satisfies to the conditions above. Then the genus of V_ζ^t is g. In addition we have $\mathcal{A}_t^\zeta = \mathrm{Log}(V_t^\zeta) \subset \mathcal{N}_\epsilon(C)$.

Proof. Recall that each point of $\partial\Delta \cap \mathbb{Z}^2$ is a vertex of Subdiv_C. If V_t^ζ satisfies to the conditions of Proposition 8.8 then no component of $V_t^\zeta \cap \mathrm{Log}^{-1}(U'(\Delta'))$ has positive genus. The condition on the number of ends of $V_t^\zeta \cap \mathrm{Log}^{-1}(U'(\Delta'))$ guarantees that the genus of V_t^ζ coincides with the genus of C. Any other choice gives a larger genus. The condition $\mathcal{A}_t^\zeta = \mathrm{Log}(V_t^\zeta) \subset \mathcal{N}_\epsilon(C)$ automatically holds for large values of t because of our constraints on ζ. □

The localization is possible thanks to the following *patchworking principle*, cf. [24]. A deformation of coefficients of monomials of index not contained in Δ' has little effect on $V_t^\zeta \cap \mathrm{Log}^{-1}(U'(\Delta'))$.

Proposition 8.9. *For any $\epsilon' > 0$ there exists $T > 1$ such that for every $t \geq T$ we have the following property. If $\zeta' \in \mathcal{D}$ is such that*

$\zeta'_j = \zeta_j$ for every $j \in \Delta'$ then $V_t^{\zeta'} \cap \operatorname{Log}^{-1}(U'(\Delta'))$ is contained in the ϵ'-neighborhood of $V_t^{\zeta} \cap \operatorname{Log}^{-1}(U'(\Delta'))$.

Proof. After an automorphism of $(\mathbb{C}^*)^2$ we may assume that $\beta_j = 0$ for $j \in \Delta'$. In this case we can take a disk around the origin for U'. In addition in this case the power of t at z^j in the expression $f_t^\zeta = \sum\limits_{j\in\Delta\cap\mathbb{Z}^2} \arg(\zeta_j) t^{-\log|\zeta_j|} z^j$ is negative if $j \notin \Delta'$. Thus for a sufficiently large t the difference of coefficients of f_t^ζ and $f_t^{\zeta'}$ is arbitrary small. Furthermore, if $\operatorname{Log}(z) \in U'$ then $f_t^\zeta(z) - f_t^{\zeta'}(z)$ can be made arbitrary small by increasing t. □

Lemma 8.10. *Let $\Delta' \subset \mathbb{R}^2$ be a triangle with vertices $k_0, k_1, k_2 \in \mathbb{Z}^2$. For any choice $b_{k_0}, b_{k_1}, b_{k_2} \in \mathbb{C}^*$ there exist $2\operatorname{Area}(\Delta')$ distinct choices of coefficients $\{b_j\}$, $j \in (\Delta \cap \mathbb{Z}^2) \smallsetminus \operatorname{Vert}(\Delta)$, such that the curve*

$$V^b = \{z \in (\mathbb{C}^*)^2 \mid \sum_{j\in\Delta\cap\mathbb{Z}^2} b_j z^j = 0\}$$

is a rational (i.e. genus 0) curve of degree Δ' with 3 ends at infinity.

Suppose that the asymptotic directions of V^b corresponding to the sides $[k_0, k_1]$ and $[k_0, k_2]$ are chosen. Out of the $2\operatorname{Area}(\Delta')$ distinct choices of coefficients above $\frac{2\operatorname{Area}(\Delta')}{l_1 l_2}$ agree with the choice of the directions, where $l_1 = \#([k_0, k_1] \cap \mathbb{Z}^2) - 1$ and $l_2 = \#([k_0, k_2] \cap \mathbb{Z}^2) - 1$.

Suppose that the asymptotic directions of the curve V^b corresponding to all three sides of Δ' are chosen then we have one or none such coefficient choices. Out of the total of $l_0 l_1 l_2$ choices of the asymptotic directions, $2\operatorname{Area}(\Delta')$ have one such choice of coefficients, $l_0 = \#([k_1, k_2] \cap \mathbb{Z}^2) - 1$.

Proof. Consider the endomorphism $\bar{M}_{\Delta'} : \mathbb{CP}^2 \to \mathbb{C}T_{\Delta'}$ of degree Δ' defined in (13). As in Lemma 7.13 any rational curve V of degree Δ' with 3 ends is an image of a line in $(\mathbb{C}^*)^2$.

Consider the closure $\bar{V}$ of V in $\mathbb{C}T_{\Delta'}$. Since $b_{k_0}, b_{k_1}, b_{k_2}$ are fixed we have l_1 possibilities for the (unique) intersection point $p_1 = \bar{V} \cap \mathbb{C}T_{[k_0,k_1]}$ and l_2 possibilities for the point $p_2 = \bar{V} \cap \mathbb{C}T_{[k_0,k_2]}$. The points p_1 and p_2 have $\frac{2\operatorname{Area}(\Delta')}{l_1}$ and $\frac{2\operatorname{Area}(\Delta')}{l_2}$ inverse images under the map $\bar{M}_{\Delta'}$. Connecting different liftings for different choices of p_1 and p_2 we get $(2\operatorname{Area}(\Delta'))^2$ different lines in $\mathbb{CP}^2$ that project to $2\operatorname{Area}(\Delta')$ different rational curves in $\mathbb{C}T_{\Delta'}$. If t is large enough then the amoeba of any such curve is contained in a small neighborhood of C □

Lemma 8.11. *Let $\Delta' \subset \mathbb{R}^2$ be a parallelogram with vertices $k_0, k_1, k_2, k_3 \in \mathbb{Z}^2$, $k_1 - k_0 = k_3 - k_2$. For any choice $b_{k_0}, b_{k_1}, b_{k_2} \in \mathbb{C}^*$ and $\epsilon' > 0$ there* *exist* $l_1 l_2 = (\#([k_0,k_1]\cap\mathbb{Z}^2)-1)(\#([k_0,k_2]\cap\mathbb{Z}^2)-1)$ *choices of coefficients* $\{b_j\}$, $j\in(\Delta'\cap\mathbb{Z}^2)\smallsetminus\{k_0,k_1,k_2\}$, *such that the curve*

$$\{z\in(\mathbb{C}^*)^2 \mid \sum_{j\in\Delta'\cap\mathbb{Z}^2} b_j z^j = 0\}$$

is a reducible curve of genus -1 *and degree* Δ' *that splits to two irreducible components of degrees* $[k_0,k_1]$ *and* $[k_0,k_2]$ *respectively.*

Suppose that the asymptotic directions of the curve corresponding to the sides $[k_0,k_1]$ *and* $[k_0,k_2]$ *are chosen. There is a unique choice of coefficients* $\{b_j\}$ *above that agrees with the choice of these directions.*

Proof. Any such curve is a union of two holomorphic cylinders taken with multiplicities l_1 and l_2 respectively. There are l_1 choices for the first cylinder and l_2 choices for the second. □

Lemma 8.12. *Let* $\Delta' = [k',k''] \in \mathrm{Subdiv}_C$ *be an edge,* $\Delta'\not\subset\partial\Delta$. *If* t *is sufficiently large then there exist* $(l')^2$ *different choices of coefficients* $\zeta'\in\mathcal{D}$ *such that* $\zeta'_j=\zeta_j$ *if* $j=k',k''$ *or* $j\in\Delta\smallsetminus\Delta'$ *and the curve*

$$\{z\in \mathrm{Log}^{-1}(U') \mid \sum_{j\in(\Delta\cap\mathbb{Z}^2)} b_j z^j = 0\}$$

is an immersed cylinder. Here $l' = \#(\Delta'\cap\mathbb{Z}^2)-1$ *is the integer length of* Δ'.

Proof. It suffices to check the lemma for a particular model. We may assume that $\Delta' = [(0,0),(l',0)]$. Let Δ be the parallelogram with vertices $(0,-1)$, $(0,0)$, $(l',0)$ and $(l',1)$. Let $g=0$ and $j_0=(0,0)$. We have $s+g-1=3$. We may choose $\mathcal{P}=\{p_1,p_2,p_3\}$ so that the only tropical curve C passing via $\mathcal{R}=\mathrm{Log}(\mathcal{P})$ has $\mathrm{Subdiv}_C \ni [(0,0),(l',0)]$ or so that $\mathrm{Subdiv}_C\ni[(0,-1),(l',1)]$. Both choices can be made so that the forest Ξ from Proposition 4.16 consists of the edges $[(0,-1),(0,0)]$, $[(0,-1),(l',0)]$ and $[(l',0),(l',1)]$. The points $\mathcal{P}$ determine the coefficients $\zeta_{(0,0)},\zeta_{(0,-1)},\zeta_{(l',0)},\zeta_{(l',1)}$ since $\partial\Delta\cap\mathbb{Z}^2$ consists only of the vertices of Δ. We need to determine the coefficients at the points $j\in\mathrm{Int}(\Delta)$. Note also that there are no lattice points inside $[(0,-1),(l',1)]$.

To compute $N^{\mathrm{irr}}(0,\Delta)$ we may use both configurations. For the first choice of $\mathcal{P}$ we have $N^{\mathrm{irr}}(g,\Delta)=N$, where N is the number of choices of ζ' we need to find. Indeed, in each triangle $T'\in\mathrm{Subdiv}_C$ we have a unique choice of coefficients with a rational curve in $\mathrm{Log}^{-1}(U'(T'))$ compatible with a chosen asymptotic direction corresponding to the side $[(0,0),(l',0)]$ by Lemma 8.10.

For the second choice of $\mathcal{P}$ we may use Lemma 8.10. We have $N^{\mathrm{irr}}(0,\Delta)=(l')^2$ and, therefore, $N=l'$. □

Lemma 8.13. *Let $\Delta' = [k', k''] \in \Xi$ be an edge. If t is sufficiently large then there exist $l' = \#(\Delta' \cap \mathbb{Z}^2) - 1$ different choices of coefficients $\zeta' \in \mathcal{D}$ such that $\zeta'_j = \zeta_j$ if $j = k', k''$ or $j \in \Delta \smallsetminus \Delta'$ and the curve*

$$\{z \in \operatorname{Log}^{-1}(U') \mid \sum_{j \in (\Delta \cap \mathbb{Z}^2)} b_j z^j = 0\}$$

is an immersed cylinder that contains a point from $\mathcal{P}$.

Proof. The proof is a small modification of the proof of the previous lemma. We just choose one of the configuration so that the edge $[(0,0),(l',0)]$ is contained in Ξ. □

We are ready to start the proof of Lemma 8.3.b. We choose the coefficients of ζ so that V_t^{ζ} passes via $\mathcal{P}$ and satisfies to the conditions of Proposition 8.8.

Recall that $C \supset \mathcal{R} \in \mathbb{R}^2$. Let $\mathcal{X} \subset \Delta$ be the tree given by Proposition 4.18. Without loss of generality we may assume that $j_0 \subset \Xi$. The number $\zeta_{j_0} = 1$ is already determined. Let us orient $\mathcal{X}$ so that j_0 is its only source. For a vertex j of $\mathcal{X}$ we denote with j' the vertex of $\mathcal{X}$ such that $[j', j]$ is a positively oriented edge of $\mathcal{X}$.

We partite the points of $(\Delta \subset \mathbb{Z}^2) \smallsetminus \{j_0\}$ into groups $G_{\Delta'} \subset \Delta \cap \mathbb{Z}^2$ corresponding to the 1- and 2-dimensional subpolygons $\Delta' \in \operatorname{Subdiv}_C$ using the following rules.

- Suppose that $\Delta' = [j', j]$ is an edge of the forest Ξ from Proposition 4.16. We let
$$G_{\Delta'} = ([j', j] \cap \mathbb{Z}^2) \smallsetminus \{j'\}.$$
- Suppose that $[j', j]$ is a diagonal of a parallelogram $\Delta' \in \operatorname{Subdiv}_C$ with the other vertices k_1, k_2. We let
$$G_{\Delta'} = (\Delta' \cap \mathbb{Z}^2) \smallsetminus ([j', k_1] \cup [j', k_2]).$$
- Suppose that $[j_1, j_2] \in \operatorname{Subdiv}_C$ is an edge such that $[j_1, j_2] \smallsetminus \{j_1, j_2\}$ is disjoint from $G_{\Delta'}$ for all parallelograms $\Delta' \subset \operatorname{Subdiv}_C$. We let
$$G_{\Delta'} = ([j_1, j_2] \cap \mathbb{Z}^2) \smallsetminus \{j_1, j_2\}.$$
- Suppose that $\Delta' \in \operatorname{Subdiv}_C$ is a triangle with vertices k_0, k_1, k_2. We let
$$G_{\Delta'} = (\operatorname{Int}(Delta') \cap \mathbb{Z}^2).$$

Clearly, the groups $G_{\Delta'}$ are disjoint. Furthermore, since $\mathcal{X}$ contains all vertices of Subdiv_C we have

$$\bigcup_{\Delta' \in \operatorname{Subdiv}_C} G_{\Delta'} = (\Delta \subset \mathbb{Z}^2) \smallsetminus \{j_0\}.$$

We say that $\zeta \in \mathcal{D}$ is Δ'-compatible in the following cases.

- If $\Delta' = [j', j]$ is an edge of the forest Ξ we require that $V_t^\zeta \cap \mathrm{Log}^{-1}(U'(\Delta'))$ is an immersed cylinder that contains a point from $\mathcal{P}$.
- If Δ' is a parallelogram we require that $V_t^\zeta \cap \mathrm{Log}^{-1}(U'(\Delta'))$ is a union of two immerse cylinders.
- If $\Delta' = [j_1, j_2]$ is an edge such that $[j_1, j_2] \smallsetminus \{j_1, j_2\}$ is disjoint from $G_{\Delta'}$ for all parallelograms $\Delta' \subset \mathrm{Subdiv}_C$ we require that $V_t^\zeta \cap \mathrm{Log}^{-1}(U'(\Delta'))$ is an immersed cylinder.
- If Δ' is a triangle we require that $V_t^\zeta \cap \mathrm{Log}^{-1}(U'(\Delta'))$ is a rational curve with 3 ends.
- If $G_{\Delta'} = \emptyset$ then any $\zeta \in \mathcal{D}$ is by default Δ'-compatible.

Lemma 8.14. *There exists an order on the polygons $\Delta' \in \mathrm{Subdiv}_C$ such that if Δ' is greater than Δ'' then $G_{\Delta'}$ is disjoint from Δ''.*

Proof. We assign the lowest order to the edges $[j_1, j_2] \in \mathrm{Subdiv}_C$ such that $[j_1, j_2] \smallsetminus \{j_1, j_2\}$ is disjoint from $G_{\Delta'}$ for all parallelograms $\Delta' \subset \mathrm{Subdiv}_C$. Their mutual order is chosen arbtrarily.

As the next step we order the edges of the tree $\mathcal{X}$ so that it agrees with the chosen orientation (recall that this is the orientation such that the only sink is j_0). Note that each such edge is either an edge Δ' of Ξ or a diagonal of a parallelogram $\Delta' \in \mathrm{Subdiv}_C$.

Finally, we assign the highest order to the triangles $\Delta' \in \mathrm{Subdiv}_C$. Again their mutual order can be chosen arbitrarily. □

Let $\Delta'_1, \dots, \Delta'_N$ be the positive-dimensional polygons from Subdiv_C enumerated according to an order given by Lemma 8.14.

Definition 8.15. The *inner multiplicity* μ'_k of a polygon Δ'_k is the following number.

- If Δ'_k is an edge then $\mu'_k = (l_k)^2$, where the integer length l_k is defines as $\#(\Delta'_k \cap \mathbb{Z}^2)$.
- If Δ'_k is a triangle then $\mu'_k = \frac{2\,\mathrm{Area}(\Delta'_k)}{l'_k}$, where l'_k is the product of the integer lengths of the 3 sides of Δ'_k. Note that in this case μ'_k is often non-integer.
- If Δ'_k is a paralellogram then $\mu'_k = 1$.

For any edge E of Subdiv_C contained entirely in $\partial\Delta$ we have $G_E = \emptyset$ since $\mathcal{P}$ is in tropically general position (and thus $x = s$ for C). Therefore $\prod_{u=1}^{N} \mu'_u$ equals to the multiplicity μ of the tropical curve $C \subset \mathbb{R}^2$. Lemma 8.3.b follows inductively from the following proposition.

Proposition 8.16. *For any $\delta > 0$ there exists $T > 1$ such that for any $t \geq T$ we have the following property.*

Suppose that $\zeta \in \mathcal{D}$ is chosen compatible with $\Delta'_1, \dots, \Delta'_{k-1}$. Unless Δ' is a triangle there exists μ'_k distinct choices of $\zeta' \in \mathcal{D}$ such that if $t \geq T$ we have the following properties.

- *The parameter ζ' is compatible with $\Delta'_1, \dots, \Delta'_k$.*
- *We have $\zeta'_j = \zeta_j$ if $j \in G_{\Delta'_u}$, $u > k$.*
- *For $u < k$ We have $V_t^{\zeta'} \cap \operatorname{Log}^{-1}(U'(\Delta'_u))$ is contained in a δ-neighborhood of $V_t^{\zeta} \cap \operatorname{Log}^{-1}(U'(\Delta'_u))$.*

If Δ'_k is a triangle then we have either one or none such choices depending on the choice of asymptotic directions of $V_t^{\zeta'} \cap \operatorname{Log}^{-1}(U'(\Delta'_k))$ corresponding to the sides Δ'_u of Δ'_k, $u < k$. We have a total of l'_k of possibilities for this choice of asymptotic direction. For $2\operatorname{Area}(\Delta')$ of these l'_k choices we have a unique choice of coefficients.

Proof. We have the coefficient $\zeta'_j = \zeta_j$ already chosen for $j \in G_{\Delta'_u}$, $u > k$.

Suppose that $j \in G_{\Delta'_k}$. Let us vary the corresponding coefficients ζ'_j within $\mathcal{D}$, i.e. within the disk $|\zeta'_j - a_j| < \epsilon'_j$ for $j \in \operatorname{Vert}(\operatorname{Subdiv}_C) \cap G_{\Delta'_k}$ while $-\log|\zeta'_j| - \beta_j < \epsilon_j$ for $j \notin \operatorname{Vert}(\operatorname{Subdiv}_C) \cap G_{\Delta'_k}$. Denote the corresponding $\#(G_{\Delta'_k})$-dimensional disk with $\mathcal{D}_k$.

Inductively, Proposition 8.16 gives $\prod_{u=1}^{k-1} \mu'_u$ choices of coefficients ζ'_j, $j \in G_{\Delta'_u}$, $u < k$. By Proposition 8.9 and Lemma 8.14 we can choose $T > 1$ so large that for any choice of

$$\eta = \{\zeta'_j\}_{j \in G_{\Delta'_k}} \in \mathcal{D}_k$$

the curve $V_t^{\zeta'} \cap \operatorname{Log}^{-1}(U'(\Delta'_u))$ is contained in a small neighborhood of $V_t^{\zeta} \cap \operatorname{Log}^{-1}(U'(\Delta'_u))$.

This yields a map

$$\Phi_k : \mathcal{D}_k \to \overline{\mathcal{M}}_{l,s},$$

where $l = \#(\operatorname{Int}\Delta \cap \mathbb{Z}^2)$ and $\overline{\mathcal{M}}_{l,s} \supset \mathcal{M}_{l,s}$ is the Deligne-Mumford compactification (see e.g. [7])

$$\Phi_k : \eta \mapsto V_t^{\zeta'}.$$

The map Φ_k is *a priori* only partially defined (for η such that $V_t^{\zeta'}$ has singularities other than nodes).

The map Φ_k is arbitrary close to the map $\Psi_k : \eta \mapsto V_t^{\zeta}$. Here ζ_j for $j \in G_{\Delta'_k}$ is defined by η and ζ_j for $j \notin G_{\Delta'_k}$ is fixed for some generic values with $\zeta \in \mathcal{D}$. The map $\Psi_k : \mathcal{D}_k \to \overline{\mathcal{M}}_{l,s}$ is globally defined and intersects the stratum of Δ'_k-compatible curves in μ'_k points by Lemma 8.10, 8.11, 8.12 and 8.13. Thus we have the same for the map Φ_k. $\square$

This finishes the proof of Lemma 8.3.b and Theorem 1.

8.4. **Real curves: proof of Theorem 3 and 6.** A signed configuration $\mathcal{R}$ determines its lift $\mathcal{P} \in (\mathbb{R}^*)^2$. Indeed, for every $r \in \mathcal{R}$ the inverse image $\mathrm{Log}^{-1}(r) \cap (\mathbb{R}^*)^2$ consists of 4 points, each in its own quadrant parameterized by the corresponding sign σ.

We detect the real curves among the μ holomorphic curves from Lemma 8.3 inductively as in the proof of Proposition 8.16. For that we suppose that $\zeta \in \mathcal{D}$ is chosen so that $\zeta_j \in \mathbb{R}$, $j \in \Delta \cap \mathbb{Z}^2$ and we need to count the number $\mu'_{k,\mathbb{R}}$ of Δ'_k-compatible choices of $\zeta'_j \in \mathbb{R}$, $j \in G_{\Delta'_k}$. Clearly, $\mu'_{k,\mathbb{R}} \equiv \mu'_k \pmod 2$ if Δ'_k is not a triangle. Thus if Δ'_k is a parallelogram then $\mu'_{k,\mathbb{R}} = \mu'_k = 1$.

If $\Delta'_k \not\subset \Xi$ is an edge then $\mu'_{k,\mathbb{R}}$ is equal to 1 or 4 (according to the parity of μ'_k). Indeed, if μ'_k is odd then out of the l' asymptotic directions of $V_t^{\zeta'} \cap \mathrm{Log}^{-1}(U'(\Delta'_k))$ there is one that corresponds to a real quadrant of $(\mathbb{R}^*)^2$.

If μ'_k is even then there are two such choices. for each of them there are two real branches in the corresponding quadrant. Two additional choices come from the choice of merging of the asymptotic directions (corresponding to the 2-dimensional polygons from Subdiv_C adjacent to Δ')

If $\Delta'_k \subset \Xi$ is an edge then $\mu'_{k,\mathbb{R}}$ is equal to 1 or 4 or 0. Indeed, if μ'_k is odd then the point p of $\mathcal{P}$ corresponding to Δ'_k sits of the real branch of $V_t^{\zeta'}$.

If μ'_k is even then $\mu'_{k,\mathbb{R}} = 0$ if the two real quadrants corresponding to the possible real asymptotic directions are disjoint from $p \in (\mathbb{R}^*)^2$. If p belongs to one of these quadrants then 2 choices come from the choice of merging of the asymptotic directions and 2 from the choice of the real branch that contains p.

Suppose that Δ'_k is a triangle. If all sides of Δ'_k have odd integer length then $\mu'_k = 1$ since only one choice of asymptotic directions of $V_t^{\zeta'} \cap \mathrm{Log}^{-1}(U'(\Delta'_k))$ is real. Otherwise we need only to consider the case when real asymptotic directions exist, i.e. $\mu'_u \neq 0$ for all $u < k$.

If only one side of Δ'_k has even integer length then $\mu'_k = 1$. In this case there are 2 distinct real choices of asymptotic directions of $V_t^{\zeta'} \cap \mathrm{Log}^{-1}(U'(\Delta'_k))$. Each of the two choices corresponds to 1 real choice of $\eta \in \mathcal{D}_k$.

If all sides of Δ'_k are even then we have 8 distinct real choices of asymptotic directions of $V_t^{\zeta'} \cap \mathrm{Log}^{-1}(U'(\Delta'_k))$. However, only 4 of them correspond to real choice of $\eta \in \mathcal{D}_k$. Thus in the last case we have $\mu'_{k,\mathbb{R}} = \frac{1}{2}$.

It remains to note that the real inner multiplicities agree with Definition 6.8. This finishes the proof of Theorem 3.

To prove Theorem 6 we note that if Subdiv_C contains an edge of an even integer length then C contributes zero to the Welschinger invariant. Indeed, for each such edge E we have two real branches of $V_t^{\zeta'} \cap \mathrm{Log}^{-1}(U'(E))$. For one choice of the merging of the asymptotic directions of the real branches we get a hyperbolic node. For the other choice we have an elliptic node. This implies that the Welschinger invariant is independent of the signs of $\mathcal{R}$.

If all edges of Subdiv_C have odd integer length then ellpitic nodes can appear only from $V_t^{\zeta'} \cap \mathrm{Log}^{-1}(U'(\Delta'))$ where Δ' is a triangle. This part of $V_t^{\zeta'}$ has a total of $\#(\Delta' \cap \mathbb{Z}^2)$ nodes. None of these nodes can be real hyperbolic since the restriction $M_{\Delta'}|_{(\mathbb{R}^*)^2} : (\mathbb{R}^*)^2 \to (\mathbb{R}^*)^2$ (see 12) is injective if all sides of Δ' have odd integer length. Therefore the multiplicity $\mathrm{mult}_V^{\mathbb{R},W}(C)$ from Definition 6.14 gives the right count for Theorem 6.

8.5. **Counting by lattice paths: proof of Theorems 2, 4 and 7.** Recall that we have a linear map $\lambda : \mathbb{R}^2 \to \mathbb{R}$ injective on $\mathbb{Z}^2$. Let $L \subset \mathbb{R}^2$ be an affine line (in the classical sense) orthogonal to the kernel of λ.

We choose a configuration $\mathcal{R} = \{r_1, \dots, r_{s+g-1} \subset L$ so that the order of r_u coincides with the linear order on L. Furthermore, we choose each r_k so that the distance from r_k to r_{k-1} is much larger than the distance from r_{k-1} to r_{k-2}. Such configuration can be chosen in a tropically general position since the slope of L is irrational.

Let $C \subset \mathbb{R}^2$ be a tropical line of genus g and degree Δ passing via $\mathcal{R}$. Let Ξ be the forest Ξ from Proposition 4.16.

Lemma 8.17. *We have $C \cap L = \mathcal{R}$.*

Proof. Let K be a component of $\Gamma \smallsetminus h^{-1}(\mathcal{R})$. Suppose that $h(K)$ intersects L at a point not from $\mathcal{R}$. One of the components of $K \smallsetminus h^{-1}(L)$ would yield a bounded graph with edges at L contained in a half-plane. Clearly such graph can not be ballanced. $\square$

Corollary 8.18. *The forest $\Xi \subset \Delta$ is a λ-increasing path that connects vertices p and q as in Theorem 2.*

Proof. By Lemma 8.17 the vertices of Ξ corrspond to the components of $L \smallsetminus \mathcal{R}$. $\square$

This corollary allows to enumerate all tropical curves of genus g and degree Δ passing via $\mathcal{P}$ by the corresponding paths. Suppose that such a path γ is chosen.

The path γ determines the slope of the edges that contain points from $\mathcal{R}$. Suppose that $E_u \ni r_u$. We need to compute the number of ways to extend these edges to a tropical curve of genus g and degree Δ. Such extensions are independent in the half-planes H_+ and H_- bounded by L.

We compute these numbers inductively. Let us extend E_1 and E_2 in H_+. Since the distance between r_1 and r_2 is the smallest if these extensions intersect in H_+ they do so before intersecting the extension of any other edge E_u. Each such intersection can give a 3-valent vertex of C. Otherwise these extension pass through the intersection point without an interaction. The first case is dual to a triangle in the dual subdivision. The second case is dual to a parallelogram. Then we consider the intersection of the result with the extension of E_3 and so on. After incorporating the extension of E_{s+g-1} the resulting curve has to have the ends orthogonal to the positive part of $\partial\Delta$, i.e. given by the path α_+.

Note that our inductive procedure agrees with the definition of the positive multiplicity of γ. Similarly, the negative multiplicity of γ agrees with a choice of extensions to H_-. Note that our procedure necessarily gives a tropical curve C of degree Δ parameterized by $h : \Gamma \to C$ so that each component of $\Gamma \setminus h^{-1}(\mathcal{R})$ is a tree with one end at infinity. Therefore, the genus of C is g.

Theorems 2, 4 and 7 follow from Theorems 1, 3 and 6 respectively.

References

[1] O. Aharony, A. Hanany, *Branes, superpotentials and superconformal fixed points*, http://arxiv.org hep-th/9704170.

[2] O. Aharony, A. Hanany, B. Kol, *Webs of (p,q) 5-branes, five dimensional field theories and grid diagrams*, http://arxiv.org hep-th/9710116.

[3] L. Caporaso, J. Harris, *Counting plane curves of any genus*, Invent. Math. **131** (1998), 345-392.

[4] Y. Eliashberg, A. Givental, H. Hofer, *Introduction to symplectic field theory,* GAFA 2000, Special Volume, Part II, 560–673.

[5] M. Forsberg, M. Passare, A. Tsikh, *Laurent determinants and arangements of hyperplane amoebas,* Advances in Math. **151** (2000), 45–70.

[6] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinski, *Discriminants, resultants and multidimensional determinants*, Birkhäuser Boston 1994.

[7] J. Harris, D. Morrison, *Moduli of curves*, Graduate Texts in Mathematics, 187. Springer-Verlag, New York, 1998.

[8] I. Itenberg, V. Kharlamov, E. Shustin, *Welschinger invariant and enumeration of real plane rational curves,* http://arxiv.org/abs/math.AG/0303378.

[9] M. M. Kapranov, *Amoebas over non-Archimedian fields*, Preprint 2000.
[10] A. G. Khovanskii, *Newton polyhedra and toric varieties* (in Russian), Funkcional. Anal. i Priložen. **11** (1977), no. 4, 56-64.
[11] M. Kontsevich, Yu. Manin, *Gromov-Witten classes, quantum cohomology and enumerative geometry*, Comm. Math. Phys. **164** (1994), 525-562.
[12] M. Kontsevich, Y. Soibelman, *Homological mirror symmetry and torus fibrations,* Symplectic geometry and mirror symmetry (Seoul, 2000), 203–263, World Sci. Publishing, River Edge, NJ, 2001.
[13] A. G. Kouchnirenko,*Polyèdres de Newton et nombres de Milnor*, Invent. Math., **32** (1976), 1 - 31.
[14] G.L. Litvinov and V.P. Maslov, *The correspondence principle for idempotent calculus and some computer applications*, in *Idempotency*, J. Gunawardena (Editor), Cambridge Univ. Press, Cambridge, 1998, 420-443.
[15] G. Mikhalkin, *Real algebraic curves, moment map and amoebas*, Ann. of Math. **151** (2000), 309 - 326.
[16] G. Mikhalkin, *Decomposition into pairs-of-pants for complex algebraic hypersurfaces*, http://arxiv.org math.GT/0205011 Preprint 2002, to appear in Topology.
[17] G. Mikhalkin, *Counting curves via lattice paths in polygons,* C. R. Math. Acad. Sci. Paris **336** (2003), no. 8, 629–634.
[18] M. Passare, H. Rullgård, *Amoebas, Monge-Ampère measures, and triangulations of the Newton polytope*. Preprint, Stockholm University, 2000.
[19] J.-E. Pin, *Tropical semirings,* Idempotency (Bristol, 1994), 50–69, Publ. Newton Inst., 11, Cambridge Univ. Press, Cambridge, 1998.
[20] Z. Ran, *Enumerative geometry of singular plane curves*, Invent. Math. **97** (1989), 447-465.
[21] J. Richter-Gebert, B. Sturmfels, T. Theobald, *First steps in tropical geometry,* http://arXiv.org math.AG/0306366.
[22] B. Sturmfels, *Solving systems of polynomial equations*, CBMS Regional Conference Series in Mathematics, AMS Providence, RI 2002.
[23] R. Vakil, *Counting curves on rational surfaces*, Manuscripta Math. **102** (2000), 53-84.
[24] O.Ya. Viro, *Gluing of plane real algebraic curves and constructions of curves of degrees* 6 *and* 7, Lecture Notes in Math., 1060 (1984) 187-200.
[25] J.-Y. Welschinger, *Invariants of real rational symplectic 4-manifolds and lower bounds in real enumerative geometry,* C. R. Math. Acad. Sci. Paris **336** (2003), no. 4, 341–344.

Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA

Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario, M5S 3G3 Canada

St. Petersburg Branch of Steklov Mathematical Institute, Fontanka 27, St. Petersburg, 191011 Russia
E-mail address: `mikha@math.toronto.edu`