Practice Midterm 3 Solutions

2 True/False

1. For an alternating series of the form $a_1 - a_2 + a_3 - a_4 + \ldots$ if $a_n > a_{n+1} > 0$ and $\lim_{n\to\infty} a_n = 0$, then the series converges.

True

2. A series that converges conditionally also converges absolutely.

False, ex) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ alternating harmonic converges conditionally but harmonic series diverges

3. A series converges absolutely in the interior of its convergence set.

True

4. You can find a Taylor series about any point x=a for any function f(x).

False, only if $f^{(n)}(a)$ is defined for all n at $x=a$

5. For an alternating series if the absolute ratio test give R=0, then the series converges conditionally.

False, converges absolutely for $R<1$

3 Free Response

Using the test of your choice determine if the following ~~sequences~~ series converge/diverge

1. $\sum_{n=1}^{\infty} \frac{3n-2}{n^2+3}$

rational terms $\Rightarrow$ limit comparison test

$$\frac{3n-2}{n^2} \approx \frac{3n}{n^2} = \frac{3}{n} = b_n$$

$$\lim_{n\to\infty} \frac{\frac{3n-2}{n^2}}{\frac{3}{n}} = \lim_{n\to\infty} \frac{(3n-2)n}{3n^2} = \lim_{n\to\infty} \frac{3n^2-2n}{3n^2}$$

$$= \lim_{n\to\infty} \frac{3-\cancel{\frac{2}{n}}^{\to 0}}{3} = 1 = L$$

$0 < L < \infty \Rightarrow a_n, b_n$ converge/diverge together

$b_n = \frac{3}{n} = 3\left(\frac{1}{n}\right) =$ harmonic series which diverges

$\Rightarrow \sum_{n=1}^{\infty} \frac{3n-2}{n^2+3}$ $\boxed{\text{diverges}}$ by Limit Comparison Test

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n}$ → Alternating Series

try absolute Ratio Test

$$|a_n| = \left|\frac{(-1)^n n}{2^n}\right| = \left|\frac{n}{2^n}\right| = \frac{n}{2^n}$$

$$|a_{n+1}| = \left|\frac{(-1)^{n+1}(n+1)}{2^{n+1}}\right| = \left|\frac{n+1}{2^{n+1}}\right| = \frac{n+1}{2^{n+1}}$$

$$R = \lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{(n+1)2^n}{n\,2^{n+1}} = \lim_{n\to\infty} \frac{(n+1)}{2n}$$

$$= \lim_{n\to\infty} \frac{1+\overset{\nearrow 0}{\cancel{\frac{1}{n}}}}{2} = \frac{1}{2} = R$$

$$R<1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n} \quad \boxed{\text{converges absolutely}}$$

Determine whether the following series conditionally converges, absolutely converges, or diverges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ $= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \text{alternating harmonic series}$$

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0 \Rightarrow \text{converges}$$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges by p-series } p=1$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} \text{ converges, but } \sum_{n=1}^{\infty} \left| \frac{\cos(n\pi)}{n} \right| \text{ diverges}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} \quad \boxed{\text{converges conditionally}}$$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^n}{n^2}$

alternating series

$$|a_n| = \left|\frac{(-1)^n 2^n}{n^2}\right| = \frac{2^n}{n^2}, \quad |a_{n+1}| = \left|\frac{(-1)^{n+1} 2^{n+1}}{(n+1)^2}\right| = \frac{2^{n+1}}{(n+1)^2}$$

$$R = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)^2}}{\frac{2^n}{n^2}} = \lim_{n\to\infty} \frac{2^{n+1} \cdot n^2}{2^n \cdot (n+1)^2} = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^2$$

$$= \lim_{n\to\infty} 2\left(\frac{1}{1+\cancel{\frac{1}{n}}^{\to 0}}\right)^2 = 2(1)^2 = 2$$

$R > 1 \Rightarrow \boxed{\text{diverges}}$

5. **Find the Maclaurin series for** $f(x) = \ln(1+x)$

$f(x) = \ln(1+x)$

$f'(x) = \dfrac{1}{1+x} = (1+x)^{-1}$

$f''(x) = -(1+x)^{-2}$

$f'''(x) = 2(1+x)^{-3}$

$f^{(4)}(x) = -6(1+x)^{-4}$

$f(0) = \ln(1) = 0$

$f'(0) = (1)^{-1} = 1$

$f''(0) = -(1)^{-2} = -1$

$f'''(0) = 2(1)^{-3} = 2$

$f^{(4)}(0) = -6(1)^{-4} = -6$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\,x^n}{n!} = f(0) + \frac{f'(0)x}{1} + \frac{f''(0)x^2}{2} + \frac{f^{(3)}(0)x^3}{6} + \cdots$$

$$= 0 + x + \frac{(-1)x^2}{2} + \frac{2x^3}{3!} + \frac{(-6)x^4}{4!} + \cdots \frac{(-1)^{n+1}(n-1)!\,x^n}{n!}$$

$$= x - \frac{x^2}{2} + \frac{2x^3}{3!} - \frac{3!\,x^4}{4!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!\,x^n}{n!}$$

$$\frac{(n-1)!}{n!} = \frac{1}{n} \qquad = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}}$$

6. Find the convergence set and the radius of convergence for the Maclaurin series found in problem 5.

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

Absolute Ratio Test

$$|a_n| = \left| \frac{(-1)^{n+1} x^n}{n} \right| = \frac{|x|^n}{n}$$

$$|a_{n+1}| = \left| \frac{(-1)^{n+2} x^{n+1}}{n+1} \right| = \frac{|x|^{n+1}}{n+1}$$

$$R = \lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{n+1}}{\frac{|x|^n}{n}} = \lim_{n\to\infty} \frac{n|x|^{n+1}}{n+1\ |x|^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)|x|$$

$$= \lim_{n\to\infty} \left(\frac{1}{1+\cancel{\frac{1}{n}}^{\,0}}\right)|x| = |x|$$

$\Rightarrow$ $R = |x| < 1$ for absolute convergence

Check $R = |x| = 1$ $\Rightarrow$ $x = \pm 1$

$x = 1$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 1^n}{n} =$ alternating harmonic, converges

$x = -1$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{n} = \sum \frac{1}{n}$

= harmonic series diverges

$\Rightarrow$ Convergence Set $=$ $\boxed{-1 < x \leq 1}$ $\boxed{\text{Radius} = 1}$

7. Using the Maclaurin series you found in problem 5 solve for the power series of $g(x) = \frac{1}{1+x}$

$$f(x) = \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

$$f'(x) = \frac{1}{1+x} = g(x) = \frac{d}{dx}\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}\right]$$

$$g(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \frac{d}{dx}[x^n]}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cancel{n} x^{n-1}}{\cancel{n}}$$

$$\boxed{g(x) = \sum_{n=1}^{\infty} (-1)^{n+1} x^{n-1}}$$

8. Find the 4th degree Taylor polynomial for $h(x) = \sinh(x)$ about the point a=1.

$$T_4(x) = \sum_{n=0}^{4} \frac{h^{(n)}(1)(x-1)^n}{n!}$$

$h(x) = \sinh(x)$ $\qquad h(1) = \sinh(1) = \dfrac{e^1 - e^{-1}}{2} = \dfrac{e - \frac{1}{e}}{2}$

$h'(x) = \cosh(x)$ $\qquad h'(1) = \cosh(1) = \dfrac{e^1 + e^{-1}}{2} = \dfrac{e + \frac{1}{e}}{2}$

$h''(x) = \sinh(x)$

$h'''(x) = \cosh(x)$ $\qquad h''(1) = h^{(4)}(1) = h(1)$

$h^{(4)}(x) = \sinh(x)$ $\qquad h'''(1) = h'(1)$

note
$$h(1) = \frac{e - \frac{1}{e}}{2} = \frac{\frac{e^2}{e} - \frac{1}{e}}{2} = \frac{\frac{e^2-1}{e}}{\frac{2}{1}} = \frac{e^2-1}{2e}$$

$$h'(1) = \frac{e + \frac{1}{e}}{2} = \frac{\frac{e^2}{e} + \frac{1}{e}}{2} = \frac{\frac{e^2+1}{e}}{\frac{2}{1}} = \frac{e^2+1}{2e}$$

$$T_4(x) = h(1) + h'(1)(x-1) + \frac{h''(1)(x-1)^2}{2} + \frac{h'''(1)(x-1)^3}{6} + \frac{h^{(4)}(1)(x-1)^4}{24}$$

$$\boxed{= \frac{e^2-1}{2e} + \frac{e^2+1}{2e}(x-1) + \left(\frac{e^2-1}{2e}\right)\frac{(x-1)^2}{2} + \left(\frac{e^2+1}{2e}\right)\frac{(x-1)^3}{6} + \left(\frac{e^2-1}{2e}\right)\frac{(x-1)^4}{24}}$$