Practice Midterm Solutions

2 True/False

1. L'Hospital's Rule can be used to solve for the limit of any sequence $\{a_n\}$.

False, only if the limit as $n \to \infty$ of $\{a_n\}$ is an indeterminant form that can be written as $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

2. Any infinite series of all positive terms will diverge.

False, $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ geometric series $a=\frac{1}{2}$, $r=\frac{1}{2}$

3. $\lim_{x\to\infty} \ln(\sqrt{x^2e^x})e^{-x^2}$ *diverges*

false, $\lim_{x\to\infty} \frac{\ln(\sqrt{x^2e^x})}{e^{x^2}} = \frac{\frac{1}{2}\ln(x^2e^x)}{e^{x^2}}$

4. The limit of an infinite sequence $\{a_n\} = f(n)$ is the same as the limit of the generating function $f(x)$ as x approaches infinity.

True, (provided that $f(x)$ is continuous)

~~5.~~ We can solve analytically (write down a formula/answer for) any integral of the form $\int \tan^m(x)\sec^n(x)dx$ where m,n are both any real numbers.

3 Free Response

Evaluate the following integrals:

1. $\int \frac{x^2-2x}{\sqrt{x-2}}dx$

$u = \sqrt{x-2}$

$u^2 = x-2 \Rightarrow x = u^2+2$

$2u\,du = dx$

$$\int \frac{(u^2+2)^2 - 2(u^2+2)}{\cancel{u}}(2\cancel{u}\,du)$$

$$= \int (u^4 + 4u^2 + \cancel{4} - 2u^2 - \cancel{4})(2\,du)$$

$$= 2\int u^4 + 2u^2\,du$$

$$= 2\left[\frac{u^5}{5} + \frac{2u^3}{3}\right] + C$$

$$= \boxed{\frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2} + C}$$

2. $\int \frac{x+4}{x^2-9}dx$ $= \displaystyle\int \frac{x+4}{(x+3)(x-3)}dx = \frac{A}{x+3} + \frac{B}{x-3}$

$$x+4 = A(x-3) + B(x+3)$$

$x = 3 \Rightarrow \quad 7 = 6B \quad\quad B = 6/7$

$x = -3 \Rightarrow \quad 1 = -6A \quad\quad A = -1/6$

$$= -\frac{1}{6}\int \frac{dx}{x+3} + \frac{6}{7}\int \frac{dx}{x-3} = \boxed{-\frac{1}{6}\ln|x+3| + \frac{6}{7}\ln|x-3| + C}$$

3. $\int_1^\infty \frac{1}{(x-1)^2}dx$ let $u = x-1$, then $x = 1 \rightarrow u = 0$

$du = dx$ $\quad x = \infty \rightarrow u = \infty$

$\displaystyle\int_0^\infty \frac{1}{u^2}\,du$ $\Rightarrow$ we divide by 0 at $u=0$ $\longrightarrow$ let $\lim\limits_{a\to 0}$

and one limit is infinite $\rightarrow$ let $\lim\limits_{b\to\infty}$

$$= \lim_{a\to 0^+},\ \lim_{b\to\infty} \int_a^b \frac{1}{u^2}\,du = \lim_{\substack{a\to 0^+ \\ b\to\infty}} \left[-\frac{1}{u}\right]_a^b$$

$$= \lim_{\substack{a\to 0^+ \\ b\to\infty}} \left(\frac{-1}{b} - \frac{-1}{a}\right) = \underset{\substack{\downarrow \\ 0}}{\frac{-1}{\infty}} + \underset{\substack{\downarrow \\ \infty}}{\frac{1}{0}} = \infty$$

$\boxed{\text{diverges}}$

Evaluate the Following Limits:

4. $\lim_{x\to 1^+} \frac{\int_1^x \sin t\,dt}{x-1} = \lim_{x\to 1^+} \frac{\int_1^x \sin(t)\,dt}{x-1} = \frac{\int_1^1 \sin(t)\,dt}{1-1} = \frac{0}{0}$

$\Rightarrow$ use L'Hôpital's Rule

First Fundamental Theorem

$$\overset{L}{=} \lim_{x\to 1^+} \frac{\frac{d}{dx}\left[\int_1^x \sin(t)\,dt\right]}{\frac{d}{dx}\left[x-1\right]} = \lim_{x\to 1^+} \frac{\sin(x)}{1}$$

$$= \boxed{\sin(1)}$$

5. $\lim_{x \to 0} x^x$

$$\lim_{x \to 0} x^x = 0^0 \quad \text{indeterminate form}$$

$$\Rightarrow \text{let } y = x^x \rightarrow \ln(y) = \ln(x^x) = x\ln(x)$$

$$\lim_{x \to 0} x\ln(x) = 0 \cdot \ln(0) = 0 \cdot -\infty \quad \text{indeterminate form}$$

$$= \lim_{x \to 0} \frac{\ln(x)}{1/x} = \frac{\infty}{\infty} \Rightarrow \text{L'Hôpital's rule}$$

$$\overset{L}{=} \lim_{x \to 0} \frac{\frac{d}{dx}\left[\ln(x)\right]}{\frac{d}{dx}\left[\frac{1}{x}\right]} = \lim_{x \to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0} \frac{-x^2}{x} = \lim_{x \to 0} -x$$

$$= 0$$

$$\Rightarrow \lim_{x \to 0} \ln(x^x) = 0$$

$$\Rightarrow \lim_{x \to 0} x^x = e^0 = \boxed{1}$$

6. $\lim\limits_{n\to\infty} a_n$ where $\{a_i\} = 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$

$$a_i = 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$$

$$= \frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$$

$$\Rightarrow \quad a_n = \frac{1}{2n-1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2n-1} = \frac{1}{2\cdot\infty - 1} = \frac{1}{\infty} = \boxed{0}$$

7. Find the limit of the corresponding series, $\sum_{i=1}^{\infty} a_i$

$$a_i = 1,\ \frac{1}{3},\ \frac{1}{5},\ \frac{1}{7},\ \ldots$$

$$\Rightarrow \quad \sum_{i=1}^{\infty} a_i = \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2n-1}$$

This look similar to a harmonic series so let us use the integral test to show it diverges

$$\int_1^{\infty} \frac{1}{2x-1} = \frac{1}{2}\int_1^{\infty} \frac{1}{x-\frac{1}{2}}\,dx = \lim_{b\to\infty} \frac{1}{2}\int_1^{b} \frac{1}{x-\frac{1}{2}}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{2}\Big[\ln\left|x-\tfrac{1}{2}\right|\Big]_1^b$$

$$= \lim_{b\to\infty} \frac{1}{2}\left(\ln\left|b-\tfrac{1}{2}\right| - \ln\left|1-\tfrac{1}{2}\right|\right)$$

$$= \frac{1}{2}\ln(\infty) - \frac{1}{2}\ln\left|\tfrac{1}{2}\right| = \infty$$

$\Rightarrow$ integral test diverges $\Rightarrow$ series $\boxed{\text{diverges}}$