Practice Midterm Solutions

2 True/False

1. The Power Rule of integration works for any integral of the form x^a where a is any real number.

   False, Power Rule fails for $x = -1$

2. We can find the slope of an inverse function even if we can't write down a formula for the inverse function.

   True (Theorem 6.2B)

3. $\sin(\sin^{-1}(3\pi)) = 3\pi$

   False, $3\pi \notin$ restricted domain

   $\sin(\sin^{-1}(3\pi)) = \sin^{-1}(\sin(3\pi)) = \sin^{-1}(0) = \boxed{0}$

4. As the step size h gets smaller (i.e we take more steps) Euler's method gets more accurate.

   True

5. We can solve analytically (write down a formula/answer for) any integral of the form $\int \tan^m(x) \sec^n(x) dx$ where m,n are both any real numbers.

   False, we need either n to be even, or m to be odd

6. $y = -4x^5 + 3x - 4$ is an invertible function.

$$\frac{dy}{dx} = -20x^4 + 3 \Rightarrow \text{False, } \frac{dy}{dx} \text{ is only strictly monotonic}$$

$\frac{dy}{dx} > 0$ for $x=0$, $\frac{dy}{dx} < 0$ for $x=1$ on a restricted domain

3 Free Response

Evaluate the Following Derivatives:

1. $\frac{d}{dx}[x^2 \ln(x) + e^{2x^2+3} - 3^x]$

$$= \frac{d}{dx}\left[x^2 \ln(x)\right] + \frac{d}{dx}\left[e^{2x^2+3}\right] - \frac{d}{dx}\left[3^x\right]$$

product rule chain rule a^x form

$$= \left(2x \ln(x) + x^2 \cdot \frac{1}{x}\right) + \left(e^{2x^2+3}(4x)\right) - 3^x \ln(3)$$

$$= \boxed{2x \ln(x) + x + 4xe^{2x^2+3} - \ln(3)\, 3^x}$$

2. $D_x[\sin(\sinh(3x))]$

chain rule 1

$= \cos(\sinh(3x)) \cdot \frac{d}{dx}[\sinh(3x)]$

$= \cos(\sinh(3x)) \cdot \cosh(3x) \cdot 3$

$= \boxed{3\cosh(3x)\cos(\sinh(3x))}$

3. find $\frac{dy}{dx}$ if $y = (x+1)^{\int_1^x (4x+3)\,dx}$

form of $f(x)^{g(x)}$ $\Rightarrow$ logarithmic differentiation

$\ln(y) = \ln\left((x+1)^{\int_1^x 4x+3\,dx}\right) = \int_1^x (4x+3)\,dx \cdot \ln(x+1)$

product rule

now differentiate both sides:

$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[\int_1^x (4x+3)\,dx\right]\ln(x+1) + \int_1^x (4x+3)\,dx\,\frac{d}{dx}[\ln(x+1)]$

F.F.T.C.

$\frac{1}{y}\frac{dy}{dx} = (4x+3)\ln(x+1) + \int_1^x (4x+3)\,dx \cdot \frac{1}{x+1}$

$\boxed{\frac{dy}{dx} = \left[(4x+3)\ln(x+1) + \frac{1}{x+1}\int_1^x (4x+3)\,dx\right](x+1)^{\int_1^x (4x+3)\,dx}}$

4. $D_x[\sin^{-1}(3x^2 - 4x + \ln(x))]$

chain rule

let $u = 3x^2 - 4x + \ln(x)$

$$D_x\left[\sin^{-1}(u)\right] = D_u\left[\sin^{-1}(u)\right] \cdot \frac{du}{dx}$$

$$= D_u\left[\sin^{-1}(u)\right] D_x\left[3x^2 - 4x + \ln(x)\right]$$

$$= \frac{1}{\sqrt{1-u^2}}\left(6x - 4 + \frac{1}{x}\right)$$

$$= \boxed{\frac{6x + \frac{1}{x} - 4}{\sqrt{1-\left(3x^2 - 4 + \frac{1}{x}\right)^2}}}$$

Evaluate the Following Integrals:

5. $\int \frac{x+1}{\sqrt{5-4x^2}}dx$ u substitution will fail ⇒ split up the integral

$$= \int \frac{x}{\sqrt{5-4x^2}}dx + \int \frac{1}{\sqrt{5-4x^2}}dx$$

u substitution form $\sin^{-1}$ form

let $u = 5-4x^2$ let $v = 2x$

$du = -8x\,dx$ $dv = 2\,dx$

$dx = \frac{du}{-8x}$ $dx = \frac{1}{2}dv$

$$= \frac{-1}{8}\int \frac{du}{\sqrt{u}} + \frac{1}{2}\int \frac{1}{\sqrt{5-v^2}}dv$$

$$= \frac{-1}{8}\int u^{-1/2}du + \frac{1}{2}\int \frac{1}{\sqrt{(\sqrt{5})^2-v^2}}dv$$

$$= -\frac{1}{8}\left(2u^{1/2}\right) + \frac{1}{2}\cdot\sin^{-1}\left(\frac{v}{\sqrt{5}}\right) + C$$

$$= \boxed{\frac{-1}{4}\sqrt{5-4x^2} + \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{5}}\right) + C}$$

6. $\int \sin(x)\sinh(\cos(x))dx$ $= -\int \underbrace{\sinh(\cos(x))}_{\sinh(u)} \underbrace{(-\sin(x))\,dx}_{du}$

let $u = \cos(x)$

$du = -\sin(x)\,dx$

$$= -\int \sinh(u)\,du$$

$$= -\left[\cosh(u) + C\right]$$

$$= \boxed{-\cosh(\cos(x)) + C}$$

7. $\int 4x^2e^x dx$ Integration by parts 2 times

$u = 4x^2$ $dv = e^x dx$

$du = 8x\,dx$ $v = e^x$

$$\Rightarrow \int 4x^2 e^x\,dx = \left[4x^2e^x\right] - \int 8xe^x\,dx$$

$u = 8x$ $dv = e^x dx$

$du = 8\,dx$ $v = e^x$

$$\Rightarrow \int 4x^2e^x\,dx = 4x^2e^x - \left(\left[8xe^x\right] - \int 8e^x\,dx\right)$$

$$= 4x^2e^x - 8xe^x + 8e^x$$

$$= e^x\left(4x^2 - 8x + 8\right) = \boxed{4e^x(x^2 - 2x + 2)}$$

8. $\int \tan^3(x)dx$ type 4 odd

$$= \int \tan^2(x)\tan(x)\,dx$$

$$= \int \left(\sec^2(x) - 1\right)\tan(x)\,dx$$

$$= \int \tan(x)\sec^2(x)\,dx - \int \tan(x)\,dx$$

$u = \tan(x)$ standard form

$du = \sec^2(x)\,dx$

$$= \int u\,du - \int \tan(x)\,dx$$

$$= \frac{1}{2}u^2 - \left(-\ln|\cos(x)| + C\right)$$

$$= \boxed{\frac{1}{2}\tan^2(x) + \ln|\cos(x)| + C}$$

alternatively

$$\int \tan(x)\sec^2(x)\,dx = \int \sec(x)\cdot\sec(x)\tan(x)\,dx$$

$u = \sec(x)$

$du = \sec(x)\tan(x)\,dx$

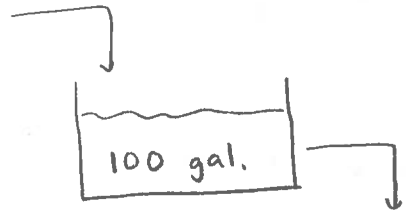
$$= \int u\,du = \frac{1}{2}u^2 = \frac{1}{2}\sec^2(x)$$

9. Salt water at a concentration of 2 pounds per gallon flows into a 100 gallon tank initially filled with fresh water (no salt) at 3 gallons per minute. If the fluid in the tank is constantly mixed and water flows out of the tank at the same rate that it flows in solve for how many pounds of salt are in the tank of water at t=10 minutes.

$* \; S = 2(1 - e^{-3t})$

$$\boxed{S(10) = 2(1 - e^{-30})}$$

$2 \frac{\text{lbs}}{\text{gal}} \cdot 3 \frac{\text{gal}}{\text{min}}$



$S \frac{\text{lbs}}{\text{gal}} \cdot 3 \frac{\text{gal}}{\text{min}}$

$$\Rightarrow \quad \frac{dS}{dt} = \text{In} - \text{Out}$$

$$= 2 \frac{\text{lbs}}{\text{gal}} \cdot 3 \frac{\text{gal}}{\text{min}} - S \frac{\text{lbs}}{\text{gal}} \cdot 3 \frac{\text{gal}}{\text{min}}$$

$$\Rightarrow \quad \frac{dS}{dt} = 6 - 3S \quad \Rightarrow \quad \frac{dS}{dt} + 3S = 6$$

first order linear differential equation

$\Rightarrow$ integrating factor technique $\quad p(t) = 3$

$$\text{I.F.} = e^{\int 3\,dt} = e^{3t}$$

$$\Rightarrow \quad \frac{d}{dt}\left[e^{3t} S\right] = 6e^{3t}$$

$$e^{3t} S = \int 6e^{3t} = 6\int e^{3t} = 6 \cdot \tfrac{1}{3} e^{3t} + C$$

$$e^{3t} S = 2e^{3t} + C$$

$$S = 2 + Ce^{-3t} \qquad S(0) = 0 = 2 + Ce^{-3\cdot 0} = C + 2$$

$$\Rightarrow C = -2$$

$$\boxed{S = 2 - 2e^{-3t}} \; *$$