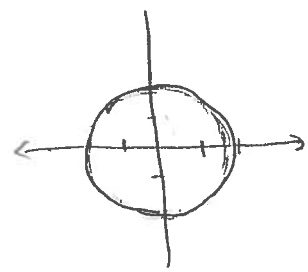
Practice Final Exam Solutions

2 True/False

1. Review the True/False questions from practice midterms 1-3 and midterms 1-3.

2. A polar coordinate function can fail the vertical line test and still be a function.

True, ex) $r = 2$

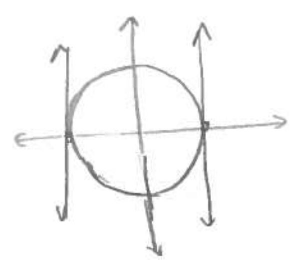


3. The origin in polar coordinates could be defined as the ordered pair $(0, 2\pi - 3)$ in polar coordinates.

True, any point (r, θ) with $r = 0$ is a way to define the origen

4. A polar coordinate function can have vertical tangent lines.

True, ex) $r = 2$



5. There are infinitely many ways to write the same coordinate location in polar coordinates.

True, ex) $(r, \theta) = (-r, \theta + \pi)$

$\parallel$

$(r, \theta + 2n\pi) = (-r, \theta + (2n+1)\pi)$

$n \in \mathbb{Z}$ (integers)

Practice Final Exam Solutions

3 Free Response

Calculate the Following Derivatives:

1. f'(x) if $f(x) = \sinh(\cos(x^2))$

2 chain Rules, let $u = \cos(x^2)$ $\quad \frac{du}{dx} = -\sin(x^2)(2x)$

$$\frac{d}{dx}\left[\sinh(u)\right] = \cosh(u)\frac{du}{dx}$$

$$= \cosh(u)\left(-\sin(x^2)(2x)\right)$$

$$= \boxed{-2x\sin(x^2)\cosh(\cos(x^2))}$$

2. $\frac{d}{dx}\left[3^{\tan(x)}\right] = \frac{d}{dx}[y] = \frac{dy}{dx}$

let $y = 3^{\tan(x)}$, then $\ln(y) = \ln\left(3^{\tan(x)}\right)$

$$\ln(y) = \tan(x)\ln(3) = \frac{\ln(3)\sin(x)}{\cos(x)}$$

$$\Rightarrow \frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[\ln(3)\frac{\sin(x)}{\cos(x)}\right] = \ln(3)\frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln(3)\left(\frac{\cos(x)\cos(x) - (-\sin(x)\sin(x))}{\cos^2(x)}\right)$$

$$\frac{dy}{dx} = y\ln(3)\left(\frac{1}{\cos^2(x)}\right) = \boxed{\ln(3)\sec^2(x)\,3^{\tan(x)}}$$

Solve the following differential equation:

3. $\dfrac{dy}{dx} = -2y,\ y(0) = 4$

$$\frac{dy}{dx} = -2y$$

$$dy = -2y\,dx$$

$$\frac{1}{y}\,dy = -2\,dx$$

$$\int \frac{1}{y}\,dy = \int -2\,dx$$

$$\ln(y) = -2x + C$$

$$e^{\ln(y)} = e^{-2x+C}$$

$$y = e^{C}e^{-2x} = Ce^{-2x}$$

$y = Ce^{-2x}$ (general solution)

$y(0) = 4 \Rightarrow 4 = Ce^{-2(0)} = C \Rightarrow C = 4$

$\boxed{y = 4e^{-2x}}$ (particular solution)

Evaluate the following integrals:

5. $\int x\sqrt{x-4}dx$

① Indefinite Integral

② u-substitution fails

③ product of functions (could use integration by parts)

④ radical of linear polynomial

Radical of Linear Polynomial

let $u=\sqrt{x-4}$

$u^2 = x-4 \Rightarrow x = u^2+4$

$2u\,du = dx$

$\Rightarrow \int x\sqrt{x-4}\,dx$

$= \int (u^2+4)(u)(2u\,du)$

$= \int (2u^4+8u^2)du$

$= \frac{2}{5}u^5 + \frac{8}{3}u^3 + C$

$= \boxed{\frac{2}{5}(x-4)^{5/2} + \frac{8}{3}(x-4)^{3/2} + C}$

Integration by parts

$\int x\sqrt{x-4}\,dx = \int u\,dv$

$\Rightarrow$ $u = x$ $\quad dv = (x-4)^{1/2}dx$

$du = dx$ $\quad v = \frac{2}{3}(x-4)^{3/2}$

$\int u\,dv = uv - \int v\,du$

$= \frac{2x}{3}(x-4)^{3/2} - \int \frac{2}{3}(x-4)^{3/2}dx$

$= \boxed{\frac{2x}{3}(x-4)^{3/2} - \frac{4}{15}(x-4)^{5/2} + C}$

note both answers could simplify

to $\frac{2}{15}(x-4)^{3/2}(3x-8) + C$

6. $\displaystyle\int_0^2 \frac{dx}{(x-1)^2}$

① Definite Integral

② No infinite bounds of integration

③ infinite integrand at $x = 1$

$$= \int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2} = \lim_{\substack{a \to 1^+ \\ b \to 1^-}} \int_0^b \frac{dx}{(x-1)^2} + \int_a^2 \frac{dx}{(x-1)^2}$$

④ u-substitution

$u = x - 1$ $\quad$ $x = 0 \to u = -1$ $\quad$ $x = a \to u = a - 1$

$du = dx$ $\quad$ $x = b \to u = b - 1$ $\quad$ $x = 2 \to u = 1$

$$= \lim_{\substack{a \to 1^+ \\ b \to 1^-}} \int_{-1}^{b-1} \frac{du}{u^2} + \int_{a-1}^{1} \frac{du}{u^2}$$

⑤ u^{-2} power rule standard form

$$= \lim_{\substack{a \to 1^+ \\ b \to 1^-}} \left[-u^{-1}\right]_{-1}^{b-1} + \left[-u^{-1}\right]_{a-1}^{1}$$

$$= \lim_{\substack{a \to 1^+ \\ b \to 1^-}} -\left(\frac{1}{b-1} - \frac{1}{-1}\right) - \left(\frac{1}{1} - \frac{1}{a-1}\right) = \boxed{\text{diverges}}$$

$\frac{1}{0^-}$ diverges $(-\infty)$ $\qquad$ $\frac{1}{0^+}$ diverge (∞)

7. $\int \tan^3(x)dx$

① Indefinite Integral

② u-sub fails

③ Trig functions $\longrightarrow$ tangent odd power

$\Rightarrow$ type 4 odd, replace $\tan^2(x)$ with $\tan^2(x) = \sec^2(x) - 1$

$$\int \tan(x)\tan^2(x)\,dx$$

$$= \int \tan(x)\left(\sec^2(x) - 1\right) dx$$

$$= \int \tan(x)\sec^2(x)\,dx - \int \tan(x)\,dx$$

$u = \tan(x)$

$du = \sec^2(x)\,dx$

$$= \int u\,du - \int \tan(x)\,dx$$

$$= \frac{1}{2}u^2 - \left(-\ln|\cos(x)|\right) + C$$

$$= \boxed{\frac{1}{2}\tan^2(x) + \ln|\cos(x)| + C}$$

8. $\int_0^{\infty} x^2 e^{-x} dx$

① Definite Integral

② Infinite Integrand

$$= \lim_{b \to \infty} \int_0^b x^2 e^{-x} \, dx$$

③ u-sub fails

④ product of fuctions $\Rightarrow$ Integration by parts

polynomial x exponential

u du

$$\Rightarrow \lim_{b \to \infty} \int_0^b x^2 e^{-x} \, dx = \lim_{b \to \infty} \left[-x^2 e^{-x} \right]_0^b - \int_0^b 2x(-e^{-x}) \, dx$$

$u = x^2$ $dv = e^{-x} dx$

$du = 2x \, dx$ $v = -e^{-x}$

$$= \lim_{b \to \infty} \left[-x^2 e^{-x} \right]_0^b + 2 \int_0^b x e^{-x} \, dx$$

$u = x$ $dv = e^{-x} dx$

$du = dx$ $v = -e^{-x}$

$$= \lim_{b \to \infty} \left[-x^2 e^{-x} \right]_0^b + 2\left[-x e^{-x} \right]_0^b - 2 \int_0^b -e^{-x} \, dx$$

$$= \lim_{b \to \infty} \left[-e^{-x} (x^2 + 2x + 2) \right]_0^b = \lim_{b \to \infty} \frac{-(b^2 + 2b + 2)}{e^b} - \frac{-2}{e^0}$$

0

by heirarchy

exponential > polynomial

$= \boxed{2}$

Evaluate the following limits:

9. $$\lim_{x\to\infty} \frac{\ln(x)}{e^x} = \frac{\infty}{\infty}$$ L'Hopitals indeterminent form

$$\overset{L}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}[e^x]} = \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x} = \lim_{x\to\infty} \frac{1}{xe^x} = \frac{1}{\infty\cdot\infty} = \boxed{0}$$

10. $$\lim_{x\to 0^+} (x+1)^{\cot(x)} = 1^{\frac{\cos(0)}{\sin(0)}} = 1^{\frac{1}{0}} = 1^{\infty}$$ indeterminant form

let $y = (x+1)^{\cot(x)} \Rightarrow \ln(y) = \cot(x)\ln(x+1)$

$$\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} \cot(x)\cdot\ln(x+1) = \infty\cdot 0$$

$$= \lim_{x\to 0^+} \frac{\ln(x+1)}{\tan(x)} = \frac{0}{0}$$ L'Hopital's indeterminent form

$$\overset{L}{=} \lim_{x\to 0^+} \frac{\left(\frac{1}{x+1}\right)}{\sec^2(x)} = \lim_{x\to 0^+} \frac{1}{(x+1)\sec^2(x)} = \lim_{x\to 0^+} \frac{\cos^2(x)}{x+1}$$

$$= \frac{1^2}{1} = 1$$

$$\Rightarrow \ln(y) = 1 \Rightarrow y = e^1 = \boxed{e}$$

Determine whether the following series converge/diverge (absolute or conditional convergence when applicable):

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^2}{n!}$

positive series, $n!$ $\Rightarrow$ Ratio Test

$$a_n = \frac{(n+1)^2}{n!} \qquad a_{n+1} = \frac{(n+2)^2}{(n+1)!}$$

$$R = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+2)^2}{(n+1)!}}{\frac{(n+1)^2}{n!}} = \lim_{n\to\infty} \frac{n!\,(n+2)^2}{(n+1)!\,(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{1}{n+1}\left(\frac{n+2}{n+1}\right)^2 = \lim_{n\to\infty} \frac{1}{n+1}\left(\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right)^2$$

$$= \frac{1}{\infty+1}\left(\frac{1+0}{1+0}\right)^2 = \frac{1}{\infty}(1)^2 = 0$$

$\Rightarrow$ $R = 0 < 1$ $\boxed{\text{converges}}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)(-1)^n}{n^2+3n}$ alternating series

no exponent, no factorial

absolute ratio test will probably give $R = 1$

$\Rightarrow$ n^{th} term test

$$\lim_{n\to\infty} \frac{(n+1)}{n^2+3n} = \lim_{n\to\infty} \frac{\frac{1}{n} + \frac{1}{n^2}}{1 + \frac{3}{n}} = \frac{0+0}{1+0} = 0$$

$$0 < \frac{n+2}{(n+1)^2 + 3(n+1)} < \frac{n+1}{n^2+3n} \Rightarrow \text{series converges}$$

Absolute or Conditional Convergence

$$\sum_{n=1}^{\infty} \left| \frac{(n+1)(-1)^n}{n^2+3n} \right| = \sum_{n=1}^{\infty} \frac{n+1}{n^2+3n} \sim \frac{n}{n^2} = \frac{1}{n} = b_n$$

Limit Comparison Test:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+1}{n^2+3n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n(n+1)}{n^2+3n}$$

$$= \lim_{n\to\infty} \frac{n^2+n}{n^2+3n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{3}{n}} = 1 \Rightarrow 0 < L < 1$$

a_n, b_n converge/diverge together, b_n = harmonic series $\Rightarrow$ diverge

alternating converge, positive diverge $\Rightarrow$ $\boxed{\text{Conditional Convergence}}$

13. **Find the 4$^{\text{th}}$ degree Taylor polynomial for $f(x) = x^2 + \ln x$ about the point x=e**

$\downarrow$

$a = e$

$f(x) = x^2 + \ln(x)$ $\quad f(e) = e^2 + 1$

$f'(x) = 2x + \frac{1}{x}$ $\quad f'(e) = 2e + \frac{1}{e}$

$f''(x) = 2 - \frac{1}{x^2}$ $\quad f''(e) = 2 - e^{-2}$

$f'''(x) = 2x^{-3}$ $\quad f'''(e) = 2e^{-3}$

$f^{(4)}(x) = -6x^{-4}$ $\quad f^{(4)}(e) = -6e^{-4}$

$$T_4(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(e)(x-e)^n}{n!}$$

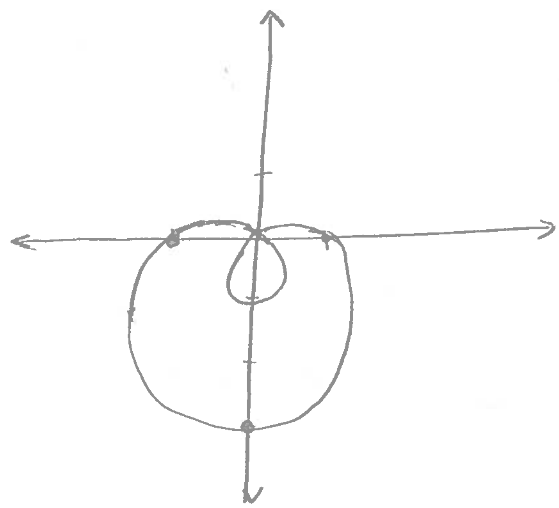
$$= \boxed{(e^2+1) + (2e + e^{-1})(x-e) + \frac{(2-e^{-2})(x-e)^2}{2} + \frac{(2e^{-3})(x-e)^3}{6} + \frac{(-6e^{-4})(x-e)^4}{24}}$$

14. For the polar coordinate equation of a limacon $r = 1 - 2\sin\theta$ sketch the graph of the equation.

Then find the area enclosed by the graph.

Finally find the slope of the line tangent to the graph at $\theta = \frac{\pi}{2}$

θ	r
0	1
$\frac{\pi}{4}$	$1-\sqrt{2} < 0$
$\frac{\pi}{3}$	$1-\sqrt{3} < 0$
$\frac{\pi}{2}$	-1
$\frac{3\pi}{4}$	$1-\sqrt{2} < 0$
$\frac{5\pi}{6}$	0
π	1
$\frac{3\pi}{2}$	3



Area (ignoring overlap w/ bulb).

$$A = \frac{1}{2}\int_0^{2\pi} (1 - 2\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (1 - 4\sin\theta + 4\sin^2\theta)\, d\theta$$

$\downarrow$ Type 1 even

$$= \frac{1}{2}\int_0^{2\pi} 1 - 4\sin\theta + 4\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 3 - 4\sin\theta - 2\cos(2\theta)\, d\theta$$

$$= \frac{1}{2}\left[3\theta + 4\cos\theta - \sin(2\theta)\right]_0^{2\pi}$$

$$= \frac{1}{2}\Big(\big(6\pi + 4 - 0\big) - \big(0 + 4 - 0\big)\Big)$$

$$= \frac{1}{2}(6\pi) = \boxed{3\pi}$$

$$\text{Slope} = \frac{f(\theta)\cos(\theta) - f'(\theta)\sin(\theta)}{-f(\theta)\sin(\theta) + f'(\theta)\cos(\theta)}$$

$r = f(\theta) = 1 - 2\sin\theta$

$f'(\theta) = -2\cos\theta$

$\theta = \frac{\pi}{2}$

$\Rightarrow \quad f\left(\frac{\pi}{2}\right) = 1 - 2\sin\left(\frac{\pi}{2}\right) = 1 - 2 = -1$

$f'\left(\frac{\pi}{2}\right) = -2\cos\left(\frac{\pi}{2}\right) = 0$

$\sin\left(\frac{\pi}{2}\right) = 1$

$\cos\left(\frac{\pi}{2}\right) = 0$

$$\text{Slope} = \frac{(-1)(0) - (0)(1)}{-(-1)(1) + (0)(0)} = \frac{0}{1} = \boxed{0}$$