Assignment 9 Solutions

Section 9.7: 1, 2, 3, 7, 13

1) $\frac{1}{1+x}$, geometric = $\frac{1}{1-x} = 1 + x + x^2 + \cdots$

$$= \sum_{n=0}^{\infty} x^n$$

$\Rightarrow$ replace x with $-x$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \boxed{\sum_{n=0}^{\infty} (-1)^n x^n}$$

$\boxed{\text{Radius of Convergence} = 1}$

2) $$\frac{1}{(1+x)^2} = \frac{d}{dx}\left[\frac{-1}{1+x}\right] = \frac{d}{dx}\left[-\sum_{n=0}^{\infty}(-1)^n x^n\right]$$

$$= -\sum_{n=0}^{\infty}(-1)^n \frac{d}{dx}[x^n] = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{d}{dx}[x^n]$$

$$= -\frac{d}{dx}[1] + \frac{d}{dx}[x] - \frac{d}{dx}[x^2] + \frac{d}{dx}[x^3] - \cdots$$

$$= 0 + 1 - 2x + 3x^2 - 4x^3 + \cdots$$

$$= \boxed{\sum_{n=0}^{\infty}(-1)^n n\, x^{n-1} \text{ , radius} = 1}$$

3) $\frac{1}{(1-x)^3}$

$$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \left[\frac{1}{(1-x)^2}\right]$$

$$\frac{d}{dx}\left[\frac{1}{(1-x)^2}\right] = \frac{d}{dx}\left[(1-x)^{-2}\right] = 2(1-x)^{-3}$$

⇒ differentiating the geometric series 2 times gives

$$\frac{2}{(1-x)^3}$$

$$\Rightarrow \frac{2}{(1-x)^3} = \frac{d}{dx}\left[\frac{d}{dx}\left[1 + x + x^2 + x^3 + \cdots\right]\right]$$

$$= \frac{d}{dx}\left[0 + 1 + 2x + 3x^2 + 4x^3 + \cdots\right]$$

$$= \left[0 + 2 + 6x + 12x^2 + \cdots\right]$$

$$\Rightarrow \frac{2}{(1-x)^3} = \sum_{n=0}^{\infty} (n+1)(n+2)x^n$$

$$\Rightarrow \frac{1}{(1-x)^3} = \boxed{\sum_{n=0}^{\infty} \frac{(n+1)(n+2)x^n}{2}, \quad \text{radius} = 1}$$

7)

$$\frac{x^2}{1-x^4} = x^2\left(\underbrace{\frac{1}{1-x^4}}_{\text{geometric, } r = x^4}\right)$$

$$= x^2\left(1 + x^4 + x^8 + x^{12} + \cdots\right)$$

$$= \boxed{x^2 + x^6 + x^{10} + x^{14} + \cdots, \quad \text{radius} = 1}$$

13) $f(x) = e^{-x}$

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!} \Rightarrow$$ replace x with $-x$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^{n}}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n}x^{n}}{n!}}$$

Section 9.8 : 1, 2, 3, 7, 13, 19, 21, 23

1) $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)}$

$$\begin{array}{r l}
 & x + \frac{x^3}{3} + \frac{2x^5}{15} \\
\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) \Big) & x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\
 & -\left(x - \frac{x^3}{2!} + \frac{x^5}{4!}\right) \\
\hline
 & \frac{x^3}{3} - \frac{4x^5}{5!} + \cdots \\
 & -\left(\frac{x^3}{3} - \frac{x^5}{6} + \cdots\right) \\
\hline
 & \frac{2x^5}{15} + \cdots
\end{array}$$

$$-\frac{x^3}{6} + \frac{x^3}{2} = \frac{x^3}{3}$$

$$\frac{x^5}{5!} - \frac{5x^5}{5!} = \frac{-4x^5}{5!}$$

$$-\frac{4x^5}{5!} + \frac{20x^5}{5!} = \frac{16x^5}{5!}$$

$$= \frac{2x^5}{15}$$

$$\boxed{\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots}$$

2) $$f(x) = \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)}{\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)}$$

$$\begin{array}{r l}
 & x - \frac{x^3}{3} + \frac{2x^5}{15} \\
\left(1 + \frac{x^2}{2} + \frac{x^4}{4!} + \cdots\right) & \overline{\Big) \; x + \frac{x^3}{6} + \frac{x^5}{5!} + \cdots} \\
 & - \left(x + \frac{x^3}{2} + \frac{x^5}{4!} + \cdots\right) \\
 & \overline{\frac{-x^3}{3} - \frac{4x^5}{5!} + \cdots} \\
 & - \left(\frac{-x^3}{3} - \frac{x^5}{6} + \cdots\right) \\
 & \overline{\frac{2x^5}{15}}
\end{array}$$

$$\frac{x^3}{6} - \frac{x^3}{2} = \frac{-x^3}{3}$$

$$\frac{x^5}{5!} - \frac{5x^5}{5!} = \frac{-4x^5}{5!}$$

$$-\frac{4x^5}{5!} + \frac{20x^5}{5!} = \frac{16x^5}{5!}$$

$$= \frac{2x^5}{15}$$

$$\boxed{\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} + \cdots}$$

3) $$f(x) = e^x \sin(x) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)$$

$$= 1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) + x\left(x - \frac{x^3}{3!} + \cdots\right)$$

$$+ \frac{x^2}{2}\left(x - \frac{x^3}{3!} + \cdots\right) + \frac{x^3}{3!}\left(x + \cdots\right) + \frac{x^4}{4!}\left(x + \cdots\right)$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + x^2 - \frac{x^4}{3!} + \frac{x^3}{2} - \frac{x^5}{2\cdot 3!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots$$

$$= x + x^2 + x^3\left(\frac{-1}{3!} + \frac{1}{2}\right) + x^4\left(\frac{-1}{3!} + \frac{1}{3!}\right) + x^5\left(\frac{1}{5!} - \frac{1}{2\cdot 3!} + \frac{1}{4!}\right)$$

$$= x + x^2 + \frac{x^3}{3} + x^5\left(\frac{1}{120} - \frac{10}{120} + \frac{5}{120}\right)$$

$$= \boxed{x + x^2 + \frac{x^3}{3} - \frac{4x^5}{5!} + \cdots}$$

7) $f(x) = e^x + x + \sin(x)$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right) + x + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$= \boxed{1 + 3x + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{2x^5}{5!} + \cdots}$$

13) $f(x) = \sin^3(x)$

$$= \left(x - \frac{x^3}{3!} + \cdots\right)^3 = \left(x - \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right)$$

$$= \left(x^2 - \frac{2x^4}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right)$$

$$= x^3 - \frac{2x^5}{3!} - \frac{x^5}{3!} + \cdots$$

$$= x^3 - \frac{3x^5}{3!} = \boxed{x^3 - \frac{x^5}{2} + \cdots}$$

19) $T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(a)(x-a)^n}{n!}$

$$= \sum_{n=0}^{3} \frac{f^{(n)}(1)(x-1)^n}{n!}$$

$$\begin{aligned} f(x) &= e^x & & & f(1) &= e \\ f'(x) &= e^x & &\Rightarrow & f'(1) &= e \\ f''(x) &= e^x & & & f''(1) &= e \\ f'''(x) &= e^x & & & f'''(1) &= e \end{aligned}$$

$$\boxed{T_3(x) = e + e(x-1) + \frac{e(x-1)^2}{2} + \frac{e(x-1)^3}{3!} + \cdots}$$

21) $\cos(x)$, $a = \frac{\pi}{3}$

$$\begin{aligned} f(x) &= \cos(x) & & & f\left(\tfrac{\pi}{3}\right) &= \cos\left(\tfrac{\pi}{3}\right) = \tfrac{1}{2} \\ f'(x) &= -\sin(x) & &\Rightarrow & f'\left(\tfrac{\pi}{3}\right) &= -\sin\left(\tfrac{\pi}{3}\right) = -\tfrac{\sqrt{3}}{2} \\ f''(x) &= -\cos(x) & & & f''\left(\tfrac{\pi}{3}\right) &= -\cos\left(\tfrac{\pi}{3}\right) = -\tfrac{1}{2} \\ f'''(x) &= \sin(x) & & & f'''\left(\tfrac{\pi}{3}\right) &= \sin\left(\tfrac{\pi}{3}\right) = \tfrac{\sqrt{3}}{2} \end{aligned}$$

$$\boxed{T_3(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{\frac{1}{2}\left(x - \frac{\pi}{3}\right)^2}{2} + \frac{\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)^3}{6}}$$

23) $1 + x^2 + x^3$, $a = 1$

$f(x) = 1 + x^2 + x^3$ $\qquad f(1) = 3$

$f'(x) = 2x + 3x^2$ $\qquad f'(1) = 5$

$f''(x) = 2 + 6x$ $\qquad f''(1) = 8$

$f'''(x) = 6$ $\qquad f'''(1) = 6$

$$T_3 = \sum_{n=0}^{3} \frac{f^{(n)}(1)(x-1)^n}{n!} = f(1) + \frac{f'(1)(x-1)}{1} + \frac{f''(1)(x-1)^2}{2} + \frac{f'''(1)(x-1)^3}{6}$$

$$= \boxed{3 + 5(x-1) + \frac{8(x-1)^2}{2} + \frac{6(x-1)^3}{6}}$$

$$= 3 + 5x - 5 + 4(x^2 - 2x + 1) + (x^3 - 3x^2 + 3x - 1)$$

$$= (3 - 5 + 4 - 1) + (5x - 8x + 3x) + (4x^2 - 3x^2) + x^3$$

$$= \boxed{1 + x^2 + x^3}$$

Section 9.9 : 1, 2, 7, 8, 11, 12, 15, 29, 31

1) $f(x) = e^{2x}$

$$f(0) = 1$$

$$f'(x) = 2e^{2x} \qquad f'(0) = 2$$

$$f''(x) = 4e^{2x} \quad \Rightarrow \quad f''(0) = 4$$

$$f'''(x) = 8e^{2x} \qquad f'''(0) = 8$$

$$f^{(4)}(x) = 16e^{2x} \qquad f^{(4)}(0) = 16$$

$$\Rightarrow \quad M_4(x) = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!}$$

$$= \boxed{1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3}}$$

$$M_4(0.12) = 1.2712 \quad , \quad f(0.12) = e^{0.24} = 1.2712$$

2) $f(x) = e^{-3x}$

$$f^{(n)}(x) = -3^n e^{-3x} \quad \Rightarrow \quad f^{(n)}(0) = -3^n$$

$$\boxed{M_4(x) = 1 - 3x + \frac{9x^2}{2} - \frac{27x^3}{6} + \frac{81x^4}{24}}$$

$$M_4(0.12) = 0.6977 \qquad f(0.12) = e^{-0.36} = 0.69767$$

7) $f(x) = \tan^{-1}(x)$

$$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

$$f''(x) = -2x(1+x^2)^{-2}$$

$$f'''(x) = 4x(1+x^2)^{-3}(2x) - 2(1+x^2)^{-2} = 8x^2(1+x^2)^{-3} - 2(1+x^2)^{-2}$$

$$f^{(4)}(x) = -24x^2(1+x^2)^{-4}(2x) + 16x(1+x^2)^{-3} + 4(1+x^2)^{-3}(2x)$$

$$= -48x^3(1+x^2)^{-4} + 8x(1+x^2)^{-3} + 16x(1+x^2)^{-3}$$

$\Rightarrow$ $f(0) = \tan^{-1}(0) = 0$, $f'(0) = 1^{-1} = 1$, $f''(0) = 0$

$f'''(0) = -2$, $f^{(4)}(0) = 0$

$$M_4(x) = 0 + \frac{1x}{1} + \frac{0x^2}{2!} - \frac{2x^3}{3!} + \frac{0x^4}{4!}$$

$$\boxed{M_4(x) = x - \frac{x^3}{3}}$$

$M_4(.12) = 0.11942$ $f(.12) = \tan^{-1}(0.12) = 0.11943$

8) $f(x) = \sinh(x)$

$$f'(x) = \cosh(x)$$
$$f''(x) = \sinh(x)$$
$$f'''(x) = \cosh(x)$$
$$f^{(4)}(x) = \sinh(x)$$

$\Rightarrow$

$$f(0) = \sinh(0) = 0$$
$$f'(0) = \cosh(0) = 1$$
$$f''(0) = f^{(4)}(0) = 0$$
$$f'''(0) = 1$$

$$M_4(x) = 0 + \frac{1 \cdot x}{1} + \frac{0x^2}{2} + \frac{1 \cdot x^3}{3!} + \frac{0x^4}{4!}$$

$$\boxed{M_4(x) = x + \frac{x^3}{3!}}$$

$M_4(0.12) = 0.1203$

$f(0.12) = \sinh(0.12) = 0.1203$

11) $\tan(x)$, $a = \frac{\pi}{6}$

$$f(x) = \tan(x)$$
$$f'(x) = \sec^2(x)$$
$$f''(x) = 2\sec(x)\left(\sec(x)\tan(x)\right) = 2\sec^2(x)\tan(x)$$
$$f'''(x) = 4\sec(x)\tan(x)\left(\sec(x)\tan(x)\right) + 2\sec^4(x)$$
$$= 4\sec^2(x)\tan^2(x) + 2\sec^4(x)$$

$$f\left(\frac{\pi}{6}\right) = \frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \tan\left(\frac{\pi}{6}\right), \quad \sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{6}\right)} = \frac{2}{\sqrt{3}}$$

$$f'\left(\frac{\pi}{6}\right) = \frac{4}{3}, \quad f''\left(\frac{\pi}{6}\right) = 2\left(\frac{2}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right) = \frac{8}{3\sqrt{3}}$$

$$f'''\left(\frac{\pi}{6}\right) = 4\left(\frac{2}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right)^2 + 2\left(\frac{2}{\sqrt{3}}\right)^4 = \frac{16}{9} + \frac{32}{9} = \frac{48}{9} = \frac{16}{3}$$

$$T_3(x) = \frac{1}{\sqrt{3}} + \frac{4}{3}\left(x-\frac{\pi}{6}\right) + \frac{8}{3\sqrt{3}}\frac{\left(x-\frac{\pi}{6}\right)^2}{2} + \frac{16}{3}\frac{\left(x-\frac{\pi}{6}\right)^3}{6}$$

$$= \boxed{\frac{\sqrt{3}}{3} + \frac{4}{3}\left(x-\frac{\pi}{6}\right) + \frac{4\sqrt{3}}{9}\left(x-\frac{\pi}{6}\right)^2 + \frac{8}{9}\left(x-\frac{\pi}{6}\right)^3}$$

12) $\sec(x)$, $a = \frac{\pi}{4}$

$$f(x) = \sec(x)$$

$$f'(x) = \sec(x)\tan(x)$$

$$f''(x) = \tan(x)\left(\sec(x)\tan(x)\right) + \sec(x)\left(\sec^2(x)\right)$$

$$= \sec(x)\tan^2(x) + \sec^3(x)$$

$$f'''(x) = \tan^2(x)\left(\sec(x)\tan(x)\right) + 2\sec(x)\tan(x)\left(\sec^2(x)\right)$$

$$+ 3\sec^2(x)\left(\sec(x)\tan(x)\right)$$

$$= \sec(x)\tan^3(x) + 2\sec^3(x)\tan(x) + 3\sec^3(x)\tan(x)$$

$$= 5\sec^3(x)\tan(x) + \sec(x)\tan^3(x)$$

note: $\sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos\left(\frac{\pi}{4}\right)} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\tan\left(\frac{\pi}{4}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$

$\Rightarrow$ $f(0) = \sqrt{2}$, $f'(0) = 1 \cdot \sqrt{2} = \sqrt{2}$, $f''(0) = \sqrt{2}(1)^2 + (\sqrt{2})^3 = \sqrt{2} + 2\sqrt{2}$

$= 3\sqrt{2}$

$$f'''(0) = 5(\sqrt{2})^3(1) + (\sqrt{2})(1)^3 = 10\sqrt{2} + \sqrt{2} = 11\sqrt{2}$$

⑫ $$\boxed{T_3(x) = \sqrt{2} + \sqrt{2}\left(x-\frac{\pi}{4}\right) + \frac{3\sqrt{2}\left(x-\frac{\pi}{4}\right)^2}{2} + \frac{11\sqrt{2}\left(x-\frac{\pi}{4}\right)^3}{6}}$$

15) $f(x) = x^3 - 2x^2 + 3x + 5 \,, \quad a = 1$

$$\begin{aligned} f'(x) &= 3x^2 - 4x + 3 \\ f''(x) &= 6x - 4 \\ f'''(x) &= 6 \end{aligned} \quad \Rightarrow \quad \begin{aligned} f(1) &= 7 \\ f'(1) &= 2 \\ f''(1) &= 2 \\ f'''(1) &= 6 \end{aligned}$$

$$\begin{aligned} T_3(x) &= 7 + 2(x-1) + \frac{2(x-1)^2}{2} + \frac{6(x-1)^3}{3!} \\ &= 7 + 2x - 2 + (x^2 - 2x + 1) + (x^3 - 3x^2 + 3x - 1) \\ &= (7 - 2 + 1 - 1) + (2x - 2x + 3x) + (x^2 - 3x^2) + x^3 \\ &= 5 + 3x - 2x^2 + x^3 = f(x) \ \checkmark \end{aligned}$$

29) $|e^{2c} + e^{-2c}| \,, \quad c \in [0,3]$

$$|e^{2c} + e^{-2c}| \le \underset{\substack{\swarrow \\ \text{growth} \\ \text{biggest for } c=3}}{|e^{2c}|} + \underset{\substack{\downarrow \\ \text{decay} \\ \text{biggest for } c=0}}{|e^{-2c}|} \le |e^{2\cdot 3}| + |e^{-2\cdot 0}| = \boxed{e^6 + 1}$$

31) $$\left|\frac{4c}{\sin(c)}\right| \le \frac{|4c|}{|\sin(c)|} \le \frac{\text{biggest}}{\text{smallest}} \qquad c \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$$

$4c$ is biggest for $c = \frac{\pi}{2}$

$\sin(c)$ is smallest for $c = \frac{\pi}{4}$

$$= \left|\frac{4\cdot\frac{\pi}{2}}{\sin\left(\frac{\pi}{4}\right)}\right| = \frac{2\pi}{\frac{\sqrt{2}}{2}} = \boxed{\frac{4\pi}{\sqrt{2}}}$$