Assignment 7 Solutions

Section 9.1: 1, 4, 9, 15, 22, 25, 30, 31, 34

1) $a_n = \dfrac{n}{3n-1}$, $a_1 = \dfrac{1}{3(1)-1} = \dfrac{1}{3-1} = \dfrac{1}{2}$

$a_2 = \dfrac{2}{3(2)-1} = \dfrac{2}{6-1} = \dfrac{2}{5}$

$a_3 = \dfrac{3}{3(3)-1} = \dfrac{3}{9-1} = \dfrac{3}{8}$

$a_4 = \dfrac{4}{3(4)-1} = \dfrac{4}{12-1} = \dfrac{4}{11}$

$a_5 = \dfrac{5}{3(5)-1} = \dfrac{5}{15-1} = \dfrac{5}{14}$

$$\lim_{n\to\infty} \frac{n}{3n-1} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{3n-1}{n}} = \lim_{n\to\infty} \frac{1}{3-\frac{1}{n}} = \frac{1}{3}$$ converges

4) $a_n = \dfrac{3n^2+2}{2n-1} \Rightarrow a_1 = 5,\ a_2 = \dfrac{14}{3},\ a_3 = \dfrac{29}{5},\ a_4 = \dfrac{50}{7}$

$a_5 = \dfrac{77}{9}$

$$\lim_{n\to\infty} \frac{3n^2+2}{2n-1} = \lim_{n\to\infty} \frac{3+\frac{2}{n^2}}{\frac{2}{n}-\frac{1}{n^2}} = 3$$

9) $a_n = \dfrac{\cos(n\pi)}{n} \Rightarrow a_1 = -1,\ a_2 = \dfrac{1}{2},\ a_3 = \dfrac{-1}{3},\ a_4 = \dfrac{1}{4}$

$a_5 = \dfrac{-1}{5}$

$$\lim_{n\to\infty} \frac{\cos(n\pi)}{n} = \lim_{n\to\infty} \frac{(-1)^n}{n}$$

note: $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left|\dfrac{(-1)^n}{n}\right| = \lim_{n\to\infty} \dfrac{1}{n} = 0$

$\Rightarrow$ By Theorem C since $\lim_{n\to\infty} |a_n| = 0$, we have

that $\lim_{n\to\infty} a_n = \boxed{0}$

15) $a_n = 2 + (0.99)^n$

$a_1 = 2.99$, $a_2 \sim 2.98$, $a_3 \sim 2.97$

$a_4 \sim 2.96$, $a_5 \sim 2.95$

$$\lim_{n\to\infty} 2 + (0.99)^n = 2 + \lim_{n\to\infty} \left(\frac{99}{100}\right)^n$$

call $y = \left(\frac{99}{100}\right)^n$, then $\ln(y) = n \cdot \ln\left(\frac{99}{100}\right)$

$$\lim_{n\to\infty} \ln(y) = \lim_{n\to\infty} n \cdot \underbrace{\ln\left(\frac{99}{100}\right)}_{<0} = -\infty$$

$$\Rightarrow \lim_{n\to\infty} y = e^{-\infty} = 0$$

$$\Rightarrow \lim_{n\to\infty} 2 + (0.99)^n = 2 + 0 = \boxed{2}$$

22) $\{a_i\} = \frac{1}{2^2}, \frac{2}{2^3}, \frac{3}{2^4}, \frac{4}{2^5}, \ldots$

top increases by 1 each time

exponent of the bottom goes up by 1 each time

$$\Rightarrow a_n = \frac{n}{2^{n+1}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2^{n+1}} = \frac{\text{polynomial}}{\text{exponential}} \Rightarrow \boxed{0}$$ (bottom grows faster)

25) $1, \frac{2}{2^2-1^2}, \frac{3}{3^2-2^2}, \frac{4}{4^2-3^2}$

$$a_n = \frac{n}{n^2-(n-1)^2} = \frac{n}{n^2-(n^2-2n+1)} = \frac{n}{2n-1}$$

$$\lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \boxed{\frac{1}{2}}$$

30) $1-\frac{1}{2}, \frac{1}{2}-\frac{1}{3}, \frac{1}{3}-\frac{1}{4}, \ldots$

$$a_n = \frac{1}{n} - \frac{1}{n+1} = \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)} = \frac{n+1-n}{n(n+1)}$$

$$= \frac{1}{n(n+1)} = \frac{1}{n^2+n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2+n} = \frac{1}{\infty^2+\infty} = \boxed{0}$$

31) $a_n = \frac{4n-3}{2^n}$ $\qquad a_1 = \frac{1}{2}, \; a_2 = \frac{5}{4}, \; a_3 = \frac{9}{8}$

$$a_4 = \frac{13}{16}$$

for $n > 2$ it seems like the sequence is decreasing

$$a_{n+1} = \frac{4(n+1)-3}{2^{n+1}} < \frac{4n-3}{2^n} = a_n$$

$$\Rightarrow \frac{4n+1}{2} < 4n-3$$

$$\Rightarrow 4n+1 < 8n-6$$

$$7 < 4n \quad \Rightarrow \quad n > \frac{7}{4}$$

so for $n > \frac{7}{4}$ $a_{n+1} < a_n$

$\Rightarrow$ the sequence is decreasing $\Rightarrow$ lets find a lower bound

$a_n > 0 \quad \forall n \geq 1 \Rightarrow 0$ is a lower bound

$\Rightarrow$ a_n decreasing but greater than 0

$\Rightarrow \lim_{n \to \infty} a_n$ converges

34) $a_n = 1 + \frac{1}{2!} + \cdots \frac{1}{n!}$

$a_1 = 1$, $a_2 = 1 + \frac{1}{2} = \frac{3}{2}$, $a_3 = 1 + \frac{1}{2} + \frac{1}{6} = \frac{5}{3}$,

$$a_4 = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = \frac{41}{24}$$

It looks like a_n is increasing

$$a_{n+1} > a_n$$

$$\sum_{i=1}^{n+1} \frac{1}{i!} > \sum_{i=1}^{n} \frac{1}{i!}$$

$$\cancel{1} + \cancel{\frac{1}{2!}} + \cancel{\frac{1}{3!}} + \cdots \cancel{\frac{1}{n!}} + \frac{1}{(n+1)!} > \cancel{1} + \cancel{\frac{1}{2!}} + \cancel{\frac{1}{3!}} + \cdots + \cancel{\frac{1}{n!}}$$

$\Rightarrow \frac{1}{(n+1)!} > 0$ true for all n

note: $a_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3 \cdot 2} + \cdots + \frac{1}{n \cdot (n-1) \cdots (3)(2)}$

$$\leq \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = b_n$$

$\Rightarrow \quad a_n \leq b_n \quad$ for all n

$b_n = \sum_{k=1}^{n} \frac{1}{2^{k-1}}$ is a geometric series

$$\Rightarrow \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = \frac{1}{1-\frac{1}{2}} = 2$$

$\Rightarrow \quad a_n$ increasing but $\leq 2 \quad \Rightarrow$ converges

Section 9.2: 1, 5, 8, 11, 15, 26, 43

1) $\sum_{k=1}^{\infty} \left(\frac{1}{7}\right)^k = \frac{1}{7} + \frac{1}{49} + \frac{1}{343} + \cdots$

note this is a geometric series, $r = \frac{1}{7}$, $a = \frac{1}{7}$

$$\sum_{k=1}^{\infty} \left(\frac{1}{7}\right)^k = \frac{a}{1-r} = \frac{1/7}{1-1/7} = \frac{1/7}{6/7} = \boxed{\frac{1}{6}}$$

5) $\sum_{k=1}^{\infty} \frac{k-5}{k+2} = \frac{-4}{3} + \frac{-3}{4} + \frac{-2}{5} + \cdots$

note $\lim_{k\to\infty} \frac{k-5}{k+2} = \lim_{k\to\infty} \frac{1-\frac{5}{k}}{1+\frac{2}{k}} = 1$

$\Rightarrow$ since $\lim_{k\to\infty} a_k \neq 0$ the series $\boxed{\text{diverges}}$

8) $\sum_{k=1}^{\infty} \frac{3}{k} = 3 + \frac{3}{2} + 1 + \frac{3}{4} + \cdots$

$= 3\sum_{k=1}^{\infty} \frac{1}{k} = 3 \times$ harmonic series $= 3 \cdot \infty = \infty$

$\boxed{\text{diverges}}$

11) $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{100^k} = \frac{1}{100} + \frac{2}{100 \cdot 100} + \cdots + \frac{101 \cdot 100 \cdot 99 \cdot \cdots 3 \cdot 2 \cdot 1}{100 \cdot 100 \cdot 100 \cdots 100 \cdot 100 \cdot 100}$

notice once k gets large enough the top starts multiplying by larger numbers than the bottom.

ie the factorial grows faster than 100^k

$$\Rightarrow \lim_{k\to\infty} \frac{k!}{100^k} = \infty \neq 0 \Rightarrow \sum_{k=1}^{\infty} \frac{k!}{100^k} \boxed{\text{diverges}}$$

15) $0.\overline{222} = 2\left(\frac{1}{10}\right) + 2\left(\frac{1}{100}\right) + 2\left(\frac{1}{1000}\right) + \cdots$

$= 2\left(\frac{1}{10}\right) + 2\left(\frac{1}{10}\right)^2 + 2\left(\frac{1}{10}\right)^3 + \cdots$ is a geometric series

$$0.\overline{222} = \sum_{n=1}^{\infty} 2\left(\frac{1}{10}\right)^n = \frac{a}{1-r}$$

$r = 1/10$

$a = 2/10$

$$= \frac{2/10}{1 - 1/10} = \frac{2/10}{9/10} = \boxed{2/9}$$

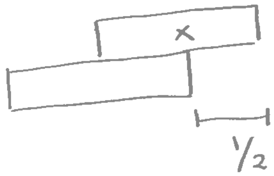
26) $\frac{1}{4} + \frac{1}{4}\left(\frac{1}{4}\right) + \frac{1}{4}\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \cdots$ geometric series

$$= \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{a}{1-r}$$

$r = \frac{1}{4}$ $a = \frac{1}{4}$

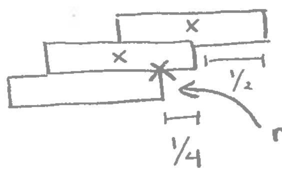
$$= \frac{1/4}{1 - 1/4} = \frac{1/4}{3/4} = \boxed{\frac{1}{3}}$$

43) a) Consider the center of masses

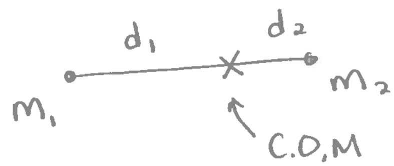


the center of mass of the first block is in the center and is supported by the right edge of the block below

now what about a 2 block system



the C.O.M of each block is in the middle so the COM of the system satisfies



$m_1 = m_2 = 1$

$$1 \cdot d_1 - 1 \cdot d_2 = 0$$

$$\Rightarrow \quad + \quad d_1 + d_2 = \frac{1}{2}$$

$$2 d_1 = \frac{1}{2}$$

$$d_1 = 1/4 = d_2$$

$$m_1 d_1 = m_2 d_2$$

$$d_1 + d_2 = 1/2$$

$\Rightarrow$ the C.O.M of the 2 block is

in general $m_1 = 1$, $m_2 = n$, $d_1 + d_2 = \frac{1}{2}$

$$\Rightarrow \quad n d_2 - d_1 = 0$$

$$+ \quad d_2 + d_1 = \frac{1}{2}$$

$$(n+1) d_2 = \frac{1}{2}$$

$$d_2 = \frac{1}{2(n+1)}$$

so the off set distances should be

$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$ as drawn

b) This total offset distance is

$$\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{2} \cdot \text{harmonic series}$$

$$= \frac{1}{2} \cdot \infty = \infty$$

$\Rightarrow$ this offset / lean can be infinite

Section 9.3 : 1, 4, 7, 13, 18, 23, 25

1) Integral test for $\sum_{k=0}^{\infty} \frac{1}{k+3}$

$$= \int_0^{\infty} \frac{1}{x+3}\,dx = \lim_{b\to\infty} \int_0^b \frac{1}{x+3}\,dx = \lim_{b\to\infty} \Big[\ln|x+3|\Big]_0^b$$

$$= \lim_{b\to\infty} \left(\ln|b+3| - \ln|3|\right)$$

$$= \ln(\infty) - \ln(3) = \infty \Rightarrow \sum_{k=0}^{\infty} \frac{1}{k+3} \text{ diverges}$$

4) $\sum_{k=1}^{\infty} \frac{k^2}{e^k} \Rightarrow \int_1^{\infty} \frac{x^2}{e^x}\,dx = \lim_{b\to\infty} \int_1^b x^2 e^{-x}\,dx$

$u = x^2 \qquad dv = e^{-x}\,dx$

$du = 2x\,dx \qquad v = -e^{-x}$

$$= \lim_{b\to\infty} \Big[-x^2 e^{-x}\Big]_1^b - \int_1^b -2x e^{-x}\,dx$$

$u = 2x \qquad dv = e^{-x}\,dx$

$du = 2\,dx \qquad v = -e^{-x}$

$$= \lim_{b\to\infty} \left[-x^2e^{-x}\right]_1^b + \left[-2xe^{-x}\right]_1^b - \int_1^b -2e^{-x}\,dx$$

$$= \lim_{b\to\infty} \left[-x^2e^{-x}\right]_1^b + \left[-2xe^{-x}\right]_1^b - 2\left[e^{-x}\right]_1^b$$

$$= \lim_{b\to\infty} \left(\underbrace{\cancel{-b^2e^{-b}}}_{\to 0} - \underbrace{\cancel{2be^{-b}}}_{\to 0} - \underbrace{\cancel{2e^{-b}}}_{\to 0} + e^{-1} + 2e^{-1} + 2e^{-1}\right)$$

note $\displaystyle \lim_{b\to\infty} \frac{b^2}{e^b} = \lim_{x\to\infty} \frac{x^2}{e^x} \overset{L}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{L}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$

$\displaystyle \lim_{b\to\infty} \frac{b}{e^b} = \lim_{x\to\infty} \frac{x}{e^x} \overset{L}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$

$= 4e^{-1} = \dfrac{4}{e} \Rightarrow$ converges $\Rightarrow \displaystyle\sum_{k=1}^{\infty} \frac{k^2}{e^k}$ converges

7) $\displaystyle\sum_{k=2}^{\infty} \frac{7}{4k+2} \Rightarrow \int_2^{\infty} \frac{7}{4x+2}\,dx$

$$= \lim_{b\to\infty} \int_2^b \frac{7}{4x+2}\,dx = \lim_{b\to\infty} \frac{7}{4}\int_2^b \frac{dx}{x+\frac{1}{2}} = \lim_{b\to\infty} \frac{7}{4}\left[\ln\left|x+\tfrac{1}{2}\right|\right]_2^b$$

$$= \lim_{b\to\infty} \frac{7}{4}\left(\ln\left|b+\tfrac{1}{2}\right| - \ln\left|2+\tfrac{1}{2}\right|\right)$$

$$= \frac{7}{4}\left(\ln(\infty) - \ln(2{,}5)\right) = \infty \quad \text{diverges}$$

$\Rightarrow \displaystyle\sum_{k=2}^{\infty} \frac{7}{4k+2}$ diverges

13) $\sum_{k=1}^{\infty} \frac{k^2+1}{k^2+5}$ $\quad a_k = \frac{k^2+1}{k^2+5}$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^2+1}{k^2+5} = \lim_{k\to\infty} \frac{1+\frac{1}{k^2}}{1+\frac{5}{k^2}} = 1 \neq 0$$

$\Rightarrow$ since $\lim_{k\to\infty} a_k \neq 0$, $\sum_{k=1}^{\infty} a_k$ $\boxed{\text{diverges}}$

18) $\sum_{k=1}^{\infty} k \sin\left(\frac{1}{k}\right)$ $\quad a_k = k\sin\left(\frac{1}{k}\right)$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} k\sin\left(\frac{1}{k}\right) = \infty \cdot \sin(0) = \infty \cdot 0$$

$$= \lim_{x\to\infty} x \sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \frac{0}{0}$$

$$\overset{L}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}\left[\sin\left(x^{-1}\right)\right]}{\frac{d}{dx}\left[x^{-1}\right]} = \lim_{x\to\infty} \frac{\cos\left(x^{-1}\right)\left(-x^{-2}\right)}{-x^{-2}}$$

$$= \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \cos\left(\frac{1}{\infty}\right) = \cos(0) = 1 \neq 0$$

$\Rightarrow$ since $\lim_{k\to\infty} a_k \neq 0$, $\sum_{k=1}^{\infty} a_k$ diverges

23) Our estimate is the sum of the first 5 terms

$$\cancel{a_1} + \cancel{a_2} + \cancel{a_3} + \cancel{a_4} + \cancel{a_5} \approx \cancel{a_1} + \cancel{a_2} + \cancel{a_3} + \cancel{a_4} + \cancel{a_5} + a_6 + \cdots$$

$\Rightarrow$ we are off by $a_6 + a_7 + a_8 + \cdots$

$$\text{error} = \sum_{k=6}^{\infty} a_k = \sum_{k=6}^{\infty} \frac{k}{e^k} \sim \int_6^{\infty} \frac{x}{e^x}\,dx$$

$$= \lim_{b\to\infty} \int_6^b x e^{-x}\,dx \qquad \begin{matrix} u = x & dv = e^{-x}dx \\ du = dx & v = -e^{-x} \end{matrix}$$

$$= \lim_{b\to\infty} \left[-xe^{-x}\right]_6^b - \int_6^b -e^{-x}\,dx$$

$$= \lim_{b\to\infty} \left[-xe^{-x}\right]_6^b - \left[e^{-x}\right]_6^b$$

$$= \lim_{b\to\infty} \left(-\cancel{be^{-b}}^{\to 0} + 6e^{-6}\right) - \left(\cancel{e^{-b}}^{\to 0} - e^{-6}\right)$$

$$= 6e^{-6} + e^{-6} = \boxed{7e^{-6}}$$

25) $$\text{error} = \sum_{k=6}^{\infty} \frac{1}{1+k^2} \sim \int_6^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{b\to\infty} \int_6^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \left[\tan^{-1}(x)\right]_6^b$$

$$= \lim_{b\to\infty} \left(\tan^{-1}(b) - \tan^{-1}(6)\right) = \tan^{-1}(\infty) - \tan^{-1}(6)$$

$$= \boxed{\frac{\pi}{2} - \tan^{-1}(6)}$$