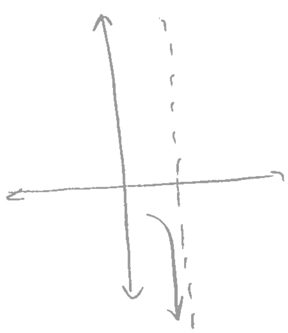
Assignment 6 Solutions

Section 8.1: 1, 7, 10, 16, 17, 19

1) $$\lim_{x\to 0} \frac{2x-\sin(x)}{x} = \frac{2(0)-\sin(0)}{0} = \frac{0}{0}$$

$$\overset{L}{=} \lim_{x\to 0} \frac{\frac{d}{dx}\left[2x-\sin(x)\right]}{\frac{d}{dx}\left[x\right]} = \lim_{x\to 0} \frac{2-\cos(x)}{1} = \frac{2-1}{1} = \boxed{1}$$

7) $$\lim_{x\to 1^-} \frac{x^2-2x+2}{x^2-1} = \frac{1^2-2(1)+2}{1^2-1} = \frac{1}{0} \Rightarrow \text{vertical asymptote at } x=1$$



lets choose a point just to the left of $x=1$ to see what direction we are asymptoting $\Rightarrow$ let $x=0.5$

$$\frac{(0.5)^2-2(0.5)+2}{(0.5)^2-1} = \frac{\frac{1}{4}-1+2}{\frac{1}{4}-1} = \frac{\frac{5}{4}}{-\frac{3}{4}} < 0$$

$$\Rightarrow \lim_{x\to 1^-} \frac{x^2-2x+2}{x^2-1} = \boxed{-\infty}$$

10) $$\lim_{x\to 0} \frac{e^x-e^{-x}}{2\sin(x)} = \frac{e^0-e^{-0}}{2\sin(0)} = \frac{1-\frac{1}{1}}{2(0)} = \frac{1-1}{0} = \frac{0}{0}$$

$$\overset{L}{=} \lim_{x\to 0} \frac{e^x-(-e^{-x})}{2\cos(x)} = \lim_{x\to 0} \frac{e^x+e^{-x}}{2\cos(x)} = \frac{e^0+e^{-0}}{2\cos(0)} = \frac{1+1}{2(1)}$$

$$= \frac{2}{2} = \boxed{1}$$

16) $$\lim_{x\to 0} \frac{\sin(x)-\tan(x)}{x^2\sin(x)} = \lim_{x\to 0}\frac{\sin(x)-\frac{\sin(x)}{\cos(x)}}{x^2\sin(x)} = \frac{0-\frac{0}{1}}{0^2\cdot 0} = \frac{0}{0}$$

$$\overset{L}{=} \lim_{x\to 0} \frac{\cos(x)-\sec^2(x)}{2x\sin(x)+x^2\cos(x)} = \lim_{x\to 0} \frac{1-\frac{1}{1^2}}{2\cdot 0\cdot 0+0^2\cdot 1} = \frac{0}{0}$$

$$\overset{L}{=} \lim_{x\to 0} \frac{-\sin(x)-2\sec^2(x)\tan(x)}{2\sin(x)+2x\cos(x)+2x\cos(x)-x^2\sin(x)} = \frac{-0-2\cdot 0}{0+0\cdot 1-0} = \frac{0}{0}$$

$$\overset{L}{=} \lim_{x\to 0} \frac{-\cos(x)-4\sec^2(x)\tan^2(x)-2\sec^4(x)}{2\cos(x)+4\cos(x)-4x\sin(x)-2x\sin(x)-x^2\cos(x)}$$

$$= \frac{-1-0-2}{6-0-0-0} = \frac{-3}{6} = \boxed{\frac{-1}{2}}$$

17) $$\lim_{x\to 0^+} \frac{x^2}{\sin(x)-x} = \frac{0^2}{0-0} = \frac{0}{0}$$

$$\overset{L}{=} \lim_{x\to 0^+} \frac{2x}{\cos(x)-1} = \frac{2\cdot 0}{1-1} = \frac{0}{0}$$

$$\overset{L}{=} \lim_{x\to 0^+} \frac{2}{-\sin(x)-0} = \lim_{x\to 0^+} \frac{2}{-\sin(x)} = \frac{2}{0}$$

from the right $-\sin(x) < 0$ $\Rightarrow$ $\lim_{x\to 0^+} \frac{x^2}{\sin(x)-x} = \boxed{-\infty}$

19) $$\lim_{x\to 0} \frac{\tan^{-1}(x)-x}{8x^3} = \frac{0-0}{8\cdot 0^3} = \frac{0}{0}$$

$$\overset{L}{=} \lim_{x\to 0} \frac{\frac{1}{1+x^2}-1}{24x^2} = \lim_{x\to 0} \frac{(1+x^2)^{-1}-1}{24x^2} = \frac{1-1}{0} = \frac{0}{0}$$

$$\overset{L}{=} \lim_{x\to 0} \frac{-2x(1+x^2)^{-2}}{48x} = \lim_{x\to 0} \frac{-2(1+x^2)^{-2}}{48} = \frac{-2}{48} = \boxed{\frac{-1}{24}}$$

Section 8.2 : 1, 3, 4, 11, 14, 19, 28, 37

1) $\lim_{x\to\infty} \frac{\ln(x^{10000})}{x} = \lim_{x\to\infty} \frac{10{,}000 \cdot \ln(x)}{x} = \frac{\infty}{\infty}$

$\overset{L}{=} \lim_{x\to\infty} \frac{10000 \cdot \frac{1}{x}}{1} = \lim_{x\to\infty} \frac{10000}{x} = \boxed{0}$

3) $\lim_{x\to\infty} \frac{x^{10000}}{e^x} = \frac{\infty}{\infty}$

$\overset{L}{=} \lim_{x\to\infty} \frac{10000\, x^{9999}}{e^x} = \frac{\infty}{\infty}$ we would have to do this a lot (10,000 times)

before we get $\overset{L}{=} \lim_{x\to\infty} \frac{10{,}000!}{e^x} = \frac{10{,}000}{\infty} = \boxed{0}$

4) $\lim_{x\to\infty} \frac{3x}{\ln(100x + e^x)} = \frac{\infty}{\infty}$

$\overset{L}{=} \lim_{x\to\infty} \frac{3}{(100+e^x)\frac{1}{100x+e^x}} = \lim_{x\to\infty} \frac{100x+e^x}{100+e^x} = \frac{\infty}{\infty}$

$\overset{L}{=} \lim_{x\to\infty} \frac{100+e^x}{e^x} = \lim_{x\to\infty} 100e^{-x} + 1 = 100 \cdot 0 + 1 = \boxed{1}$

11) $\lim_{x\to 0} x\ln(x^{1000}) = 0 \cdot \ln(0^{1000}) = 0 \cdot \ln(0) = 0 \cdot (-\infty)$

$= \lim_{x\to 0} \frac{\ln(x^{1000})}{1/x} = \lim_{x\to 0} \frac{1000 \ln(x)}{1/x} = \frac{\infty}{\infty}$

$\overset{L}{=} \lim_{x\to 0} \frac{1000 \cdot \frac{1}{x}}{-x^{-2}} = \lim_{x\to 0} \frac{\frac{1000}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0} \frac{-1000x^2}{x} = \lim_{x\to 0} -1000x$

$= \boxed{0}$

14) $\lim_{x \to \pi/2} \tan(x) - \sec(x) = \infty - \infty$

$$= \lim_{x \to \pi/2} \frac{\sin(x)}{\cos(x)} - \frac{1}{\cos(x)} = \lim_{x \to \pi/2} \frac{\sin(x) - 1}{\cos(x)} = \frac{1-1}{0} = \frac{0}{0}$$

$$\overset{L}{=} \lim_{x \to \pi/2} \frac{\cos(x)}{-\sin(x)} = \frac{0}{1} = \boxed{0}$$

19) $\lim_{x \to 0} \left(x + e^{x/3}\right)^{3/x} = \left(0 + e^0\right)^{3/0} = 1^\infty$

$$\Rightarrow \text{let } y = \left(x + e^{x/3}\right)^{3/x} \Rightarrow \ln(y) = \frac{3}{x} \ln\left(x + e^{x/3}\right)$$

$$\lim_{x \to 0} \ln(y) = \lim_{x \to 0} \frac{3 \ln\left(x + e^{x/3}\right)}{x} = \frac{3 \cdot \ln(1)}{0} = \frac{0}{0}$$

$$\overset{L}{=} \lim_{x \to 0} \frac{3 \cdot \left(1 + \frac{1}{3} e^{x/3}\right) \cdot \frac{1}{x + e^{x/3}}}{1} = \lim_{x \to 0} \frac{3\left(1 + \frac{1}{3} e^{x/3}\right)}{x + e^{x/3}}$$

$$= \frac{3\left(1 + \frac{1}{3} e^0\right)}{0 + e^0} = \frac{3\left(1 + \frac{1}{3}\right)}{1} = \frac{3\left(\frac{4}{3}\right)}{1} = 4$$

$$\Rightarrow \lim_{x \to \frac{\pi}{2}} \ln(y) = 4 \Rightarrow \lim_{x \to \frac{\pi}{2}} y = \boxed{e^4}$$

28) $\lim_{x \to 0} \left(\cos(x) - \sin(x)\right)^{1/x} = 1^\infty$

$$\Rightarrow y = \left(\cos(x) - \sin(x)\right)^{1/x} \Rightarrow \ln(y) = \frac{1}{x} \ln\left(\cos(x) - \sin(x)\right)$$

$$\lim_{x \to 0} \frac{\ln(\cos(x) - \sin(x))}{x} = \frac{\ln(1)}{0} = \frac{0}{0}$$

$$\overset{L}{=} \lim_{x\to 0} \frac{(-\sin(x)-\cos(x))\cdot \frac{1}{\cos(x)-\sin(x)}}{1} = \lim_{x\to 0} \frac{-(\sin(x)+\cos(x))}{\cos(x)-\sin(x)}$$

$$= \frac{-(0+1)}{1-0} = \frac{-1}{1} = -1$$

$$\Rightarrow \lim_{x\to 0} \ln(y) = -1 \quad \Rightarrow \lim_{x\to 0} y = e^{-1} = \boxed{\frac{1}{e}}$$

37) $\lim_{x\to 0^+} \frac{x}{\ln(x)} = \frac{0}{-\infty} = \boxed{0}$

Section 8.3: 1, 4, 7, 9, 14, 19

1) $\int_{100}^{\infty} e^x\,dx = \lim_{b\to\infty} \int_{100}^{b} e^x\,dx = \lim_{b\to\infty} \left[e^x\right]_{100}^{b}$

$$= \lim_{b\to\infty} \left(e^b - e^{100}\right) = \infty \quad \boxed{\text{diverges}}$$

4) $\int_{-\infty}^{1} e^{4x}\,dx = \lim_{a\to-\infty} \int_{a}^{1} e^{4x}\,dx = \lim_{a\to-\infty} \left[\frac{1}{4}e^{4x}\right]_a^1$

$$= \lim_{a\to-\infty} \left(\frac{1}{4}e^4 - \frac{1}{4}e^{4a}\right)$$

$$= \frac{1}{4}e^4 - \cancel{\frac{1}{4}e^{-\infty}}^{\to 0} = \boxed{\frac{1}{4}e^4}$$

7) $\int_{1}^{\infty} \frac{1}{x^{1.00001}}\,dx = \lim_{b\to\infty} \int_{1}^{b} x^{-1.00001}\,dx$

$$= \lim_{b\to\infty} \left[\frac{x^{-0.00001}}{-0.00001}\right]_1^b = \lim_{b\to\infty} \left(\cancel{\frac{b^{-0.00001}}{-0.00001}}^{\to 0} - \frac{1^{-0.00001}}{-0.00001}\right)$$

$$= -\frac{-1}{0.00001} = \boxed{100{,}000}$$

9) $$\int_1^\infty \frac{1}{x^{0.99999}}\,dx = \int_1^\infty x^{-0.99999}\,dx$$

$$= \lim_{b\to\infty} \int_1^b x^{-0.99999}\,dx = \lim_{b\to\infty} \left[\frac{x^{0.00001}}{0.00001}\right]_1^b$$

$$= \lim_{b\to\infty} \left(\frac{b^{0.00001}}{0.00001} - \frac{1^{0.00001}}{0.00001}\right)$$

$$= \infty - 100{,}000 = \infty \quad \text{diverges}$$

14) $$\int_1^\infty xe^{-x}\,dx = \lim_{b\to\infty} \int_1^b xe^{-x}\,dx$$

$$u = x \qquad dv = e^{-x}\,dx$$

$$du = dx \qquad v = -e^{-x}$$

$$= \lim_{b\to\infty} \left[-xe^{-x}\right]_1^b - \int_1^b -e^{-x}\,dx$$

$$= \lim_{b\to\infty} \left[-xe^{-x}\right]_1^b + \left[-e^{-x}\right]_1^b$$

$$= \lim_{b\to\infty} \left(-be^{-b} - (-1e^{-1})\right) + -e^{-b} - (-e^{-1})$$

$$= \lim_{b\to\infty} \left(-be^{-b} + e^{-1} - e^{-b} + e^{-1}\right)$$

$$= \infty \cdot 0 + 2e^{-1} - 0$$

$\Rightarrow$ L'Hopital's rule for $\lim_{b\to\infty} -be^{-b} = \lim_{b\to\infty} \frac{-b}{e^b} \overset{L}{=} \frac{1}{e^b} = 0$

$$\Rightarrow \int_1^\infty xe^{-x}\,dx = 2e^{-1} = \boxed{\frac{2}{e}}$$

19) $$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+10}\,dx = \int_{-\infty}^{0} \frac{1}{x^2+2x+10}\,dx + \int_{0}^{\infty} \frac{1}{x^2+2x+0}\,dx$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{x^2+2x+10}\,dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{x^2+2x+10}\,dx$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{(x+1)^2+9}\,dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{(x+1)^2+9}\,dx$$

let $u = x+1$ $\qquad x = a \quad u = a+1$

$du = dx$ $\qquad x = 0 \quad u = 1$

$\qquad x = b \quad u = b+1$

$$= \lim_{a\to-\infty} \int_{a+1}^{1} \frac{1}{u^2+9}\,du + \lim_{b\to\infty} \int_{1}^{b+1} \frac{1}{u^2+9}\,du$$

trig inverse $\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)$ standard form

$$= \lim_{a\to-\infty} \left[\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)\right]_{a+1}^{1} + \lim_{b\to\infty} \left[\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)\right]_{1}^{b+1}$$

$$= \cancel{\frac{1}{3}\tan^{-1}\left(\frac{1}{3}\right)} - \frac{1}{3}\tan^{-1}(-\infty) + \frac{1}{3}\tan^{-1}(\infty) - \cancel{\frac{1}{3}\tan^{-1}\left(\frac{1}{3}\right)}$$

$$= \frac{1}{3}\tan^{-1}(\infty) - \frac{1}{3}\tan^{-1}(-\infty)$$

$$= \frac{1}{3}\left(\frac{\pi}{2}\right) - \frac{1}{3}\left(\frac{-\pi}{2}\right) = \boxed{\frac{\pi}{3}}$$

Section 8.4 : 1, 5, 8, 11, 24, 29

1) $\int_1^3 \frac{dx}{(x-1)^{1/3}}$ note: at $x=1$ we divide by 0

$\Rightarrow$ let us replace $x=1$ with $x=a$ and take the limit as $a \to 1^+$ (from the right b/c left endpoint)

$$= \lim_{a\to 1^+} \int_a^3 \frac{1}{(x-1)^{1/3}}\,dx \quad , \text{let } u = x-1, \; du = dx \qquad x=a \to u=a-1, \; x=3 \to u=3-1=2$$

$$= \lim_{a\to 1^+} \int_{a-1}^{2} \frac{1}{u^{1/3}}\,du = \lim_{a\to 1^+} \int_{a-1}^{2} u^{-1/3}\,du$$

$$= \lim_{a\to 1^+} \left[\frac{u^{2/3}}{2/3} \right]_{a-1}^{2}$$

$$= \lim_{a\to 1^+} \left(\frac{3\cdot 2^{2/3}}{2} - \frac{3(a-1)^{2/3}}{2} \right)$$

$$= 3\cdot 2^{-1/3} - \cancel{\frac{3}{2}(0)^{2/3}} \to 0$$

$$= \boxed{\frac{3}{\sqrt[3]{2}} = \frac{3}{2}\sqrt[3]{4}}$$

5) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ $\Rightarrow$ $x=1$ we divide by 0

$$= \lim_{b\to 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1^-} \left[\sin^{-1}(x) \right]_0^b$$

$$= \lim_{b\to 1^-} \sin^{-1}(b) - \sin^{-1}(0)$$

$$= \sin^{-1}(1) - \sin^{-1}(0)$$

$$= \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}$$

8) $\int_5^{-5} \frac{1}{x^{2/3}}\,dx = -\int_{-5}^{5} \frac{1}{x^{2/3}}\,dx$ $\Rightarrow$ note that at $x=0$ we divide by 0
this happens in the interior

$$= -\int_{-5}^{0} \frac{1}{x^{2/3}}\,dx - \int_0^5 \frac{1}{x^{2/3}}\,dx$$

$$= \lim_{b\to 0^-} -\int_{-5}^{b} \frac{1}{x^{2/3}}\,dx - \lim_{a\to 0^+} \int_a^5 \frac{1}{x^{2/3}}\,dx$$

$$= \lim_{b\to 0^-} -\left[\frac{x^{1/3}}{1/3} \right]_{-5}^{b} - \lim_{a\to 0^+} \left[\frac{x^{1/3}}{1/3} \right]_a^5$$

$$= -\lim_{b\to 0^-}\left(3b^{1/3} - 3(-5)^{1/3}\right) - \lim_{a\to 0^+}\left(3(5)^{1/3} - 3a^{1/3}\right)$$

$$= -\cancel{3(0)^{1/3}} \xrightarrow{} 0 + 3(-5)^{1/3} - 3(5)^{1/3} + \cancel{3(0)^{1/3}} \xrightarrow{} 0$$

$$= 3(-5)^{1/3} - 3(5)^{1/3}$$

assume we are interested in the real root

$$= -3(5)^{1/3} - 3(5)^{1/3}$$

$$= \boxed{-6\sqrt[3]{5}}$$

11) $\displaystyle\int_0^4 \frac{dx}{(2-3x)^{1/3}}$ note for $x = \frac{2}{3}$ we divide by 0 (infinite integrand)

$$= \int_0^{2/3} \frac{dx}{(2-3x)^{1/3}} + \int_{2/3}^4 \frac{dx}{(2-3x)^{1/3}}$$

now we go to infinity at the endpoints

$$= \lim_{b\to 2/3^-} \int_0^b \frac{dx}{(2-3x)^{1/3}} + \lim_{a\to \frac{2}{3}^+} \int_a^4 \frac{dx}{(2-3x)^{1/3}}$$

let

$u = 2-3x$

$du = -3\,dx$

$\Rightarrow$

$x = 0 \longrightarrow u = 2$

$x = b \longrightarrow u = 2-3b$

$x = a \longrightarrow u = 2-3a$

$x = 4 \longrightarrow u = -10$

$$= \lim_{b \to \frac{2}{3}^-} \frac{-1}{3} \int_{2}^{2-3b} \frac{du}{u^{1/3}} + \lim_{a \to \frac{2}{3}^+} \frac{-1}{3} \int_{2-3a}^{-10} \frac{du}{u^{1/3}}$$

$$= \lim_{b \to \frac{2}{3}^-} \frac{-1}{3} \left[\frac{u^{2/3}}{2/3} \right]_{2}^{2-3b} + \lim_{a \to \frac{2}{3}^+} \frac{-1}{3} \left[\frac{u^{2/3}}{2/3} \right]_{2-3a}^{-10}$$

$$= \lim_{b \to \frac{2}{3}^-} -\frac{1}{2}\left(\overset{0}{\cancel{(2-3b)^{2/3}}} - 2^{2/3} \right) + \lim_{a \to \frac{2}{3}^+} \frac{-1}{2}\left((-10)^{2/3} - \overset{0}{\cancel{(2-3a)^{2/3}}} \right)$$

$$= \frac{1}{2}(2)^{2/3} - \frac{1}{2}(-10)^{2/3} = \boxed{\frac{1}{\sqrt[3]{2}} - \frac{\sqrt[3]{100}}{2}}$$

24) $$\int_0^{\pi/4} \frac{\sec^2(x)}{(\tan(x)-1)^2}\,dx$$

$\sec = \dfrac{1}{\cos(x)}$ $\quad\cos(x) = 0$ at $x = \frac{\pi}{2}$

$\tan(x) - 1 = 0$ when $x = \frac{\pi}{4}$

$$= \lim_{b \to \frac{\pi}{4}^-} \int_0^{b} \frac{\sec^2(x)}{(\tan(x)-1)^2}\,dx$$

let $u = \tan(x) - 1$

$du = \sec^2(x)\,dx$

$x = 0 \longrightarrow u = \tan(0) - 1 = -1$

$x = b \longrightarrow u = \tan(b) - 1$

$$= \lim_{b \to \frac{\pi}{4}^-} \int_{-1}^{\tan(b)-1} \frac{1}{u^2}\,du = \lim_{b \to \frac{\pi}{4}^-} \left[-u^{-1} \right]_{-1}^{\tan(b)-1}$$

$$= \lim_{b \to \frac{\pi}{4}^-} \left(\frac{-1}{\tan(b)-1} - \frac{-1}{-1} \right)$$

diverges

$$= \lim_{b \to \frac{\pi}{4}^-} -\frac{1}{\tan(b)-1} - 1 = \frac{1}{0} - 1 = \infty$$

29) $\displaystyle\int_1^e \frac{dx}{x\ln(x)}$ for $x=1$ $\ln(1)=0 \Rightarrow$ divide by 0

$$= \lim_{a\to 1^+} \int_a^e \frac{1}{x}\cdot\frac{1}{\ln(x)}\,dx$$

let $u=\ln(x)$

$du=\frac{1}{x}dx$

$x=a \longrightarrow u=\ln(a)$

$x=e \longrightarrow u=\ln(e)=1$

$$= \lim_{a\to 1^+}\int_{\ln(a)}^{1}\frac{1}{u}\,du = \lim_{a\to 1^+}\Big[\ln(u)\Big]_{\ln(a)}^{1}$$

$$= \lim_{a\to 1^+}\left(\cancel{\ln(1)}^{\to 0} - \ln(\ln(a))\right)$$

$$= -\ln(\ln(1^+)) = -\ln(0^+) = -(-\infty)$$

$$= \infty \quad \boxed{\text{diverges}}$$