Math 1220-006 Assignment 2

Section 6.5: 1, 2, 5, 18, 21, 25

1) $\frac{dy}{dt} = -6y$ $\quad y(0) = 4$ $\quad$ Seperable

$$\int \frac{dy}{y} = \int -6\,dt$$

$$\ln(y) = -6t + C$$

$$\Rightarrow \underbrace{e^{\ln(y)}}_{= y} = e^{-6t+C} = e^{c}e^{-6t}$$

$$\Rightarrow y = ke^{-6t} \qquad y(0) = 4$$

$$\Rightarrow 4 = ke^{-6\cdot 0} = k \Rightarrow k = 4$$

$$\boxed{y = 4e^{-6t}}$$

2) $\frac{dy}{dt} = 6y$, $y(0) = 1$ $\quad$ Separable

$$\frac{dy}{y} = 6\,dt \Rightarrow \int \frac{dy}{y} = \int 6\,dt \Rightarrow \ln(y) = 6t + C$$

$$\Rightarrow e^{\ln(y)} = e^{6t+C} \Rightarrow y = e^{c}e^{6t} = ke^{6t}$$

$$y(0) = 1 \rightarrow 1 = ke^{6\cdot 0} = k \Rightarrow k = 1$$

$$\boxed{y = e^{6t}}$$

5) $P(0) = 10{,}000$

$P(10) = 20{,}000$

let $P(t) = P_0 e^{kt}$ exponential growth

then $P(0) = P_0 e^{0 \cdot t} = P_0 = 10{,}000$

$P(10) = 10{,}000\, e^{k \cdot 10} = 20{,}000$

$\Rightarrow e^{10k} = 2$

$\ln(e^{10k}) = \ln(2)$

$10k = \ln(2) \Rightarrow k = \frac{\ln(2)}{10}$

$P(25) = 10{,}000\, e^{\frac{\ln(2)}{10}(25)} = 10{,}000\, e^{\frac{5}{2}\ln(2)}$

$= 10{,}000\, e^{\ln(2^{\frac{5}{2}})} = 10{,}000\,(2)^{5/2}$

$= 10{,}000\sqrt{32} = 40{,}000\sqrt{2} \approx \boxed{56569 \text{ bacteria}}$

18) only 51% carbon 14 remaining

half-life of carbon 14 = 5730 years

$\Rightarrow P(t) = P_0 e^{kt}$

$\frac{1}{2}\cancel{P_0} = \cancel{P_0} e^{k\,5730}$

$e^{5730k} = \frac{1}{2}$

$\ln(e^{5730k}) = \ln(\frac{1}{2}) \Rightarrow k = \frac{\ln(1/2)}{5730} \approx -0.00012$

$$\Rightarrow \quad P(t) = P_0 e^{-0.00012t}$$

if 51% remain then $P(t) = .51 P_0$

$$\Rightarrow \quad 0.51 \cancel{P_0} = \cancel{P_0} e^{-0.00012t}$$

$$e^{-0.00012t} = 0.51$$

$$-0.00012t = \ln(.51)$$

$$t = \frac{\ln(0.51)}{-0.00012} = \boxed{5566 \text{ years}}$$

21) Temp at $t=0$ is $26°C$

$T_R = 90°C$

Temp at $t=5$ is $70°C$

Newton's law of cooling/heating

$$\Rightarrow \quad \frac{dT}{dt} = k(T - T_R)$$ separable and has solution

$$\ln|T - T_R| = kt + C$$

$$\Rightarrow \quad \ln|T - 90| = kt + C$$

to deal with absolute value we need $90 - T$

$$\ln(90 - T) = kt + C$$

$$e^{\ln(90-T)} = e^{kt+C} = e^{C} e^{kt} = Ce^{kt}$$

$$\Rightarrow \quad 90 - T = Ce^{kt}$$

$$T = 90 - Ce^{kt}$$

$$T(0) = 90 - Ce^{k \cdot 0} = 90 - C = 26$$

$$\Rightarrow \quad C = 64$$

$$T(5) = 90 - 64e^{k \cdot 5} = 70$$

$$-64e^{5k} = -20$$

$$e^{5k} = \frac{20}{64}$$

$$5k = \ln\left(\frac{20}{64}\right) \quad \Rightarrow \quad k = \frac{1}{5}\ln\left(\frac{20}{64}\right)$$

$$T(10) = 90 - 64e^{\frac{1}{5}\ln\left(\frac{20}{64}\right) \cdot 10}$$

$$= 90 - 64e^{2\ln\left(\frac{20}{64}\right)} = 90 - 64e^{\ln\left(\left(\frac{20}{64}\right)^2\right)}$$

$$= 90 - 64\left(\frac{20}{64}\right)^2 = 90 - \frac{20^2}{64} = 90 - \frac{400}{64}$$

$$= \frac{335}{64} = \boxed{83.75\ {}^\circ C}$$

25) Initial investment = \$375, yearly interest = 3.5%,

2 year investment

$$\Rightarrow \quad A(t) = A_0\left(1+\frac{r}{n}\right)^{nt}$$

a) Annually $(n=1)$

$$A(2) = 375\left(1+\frac{0.035}{1}\right)^{1\cdot 2}$$

$$= 375(1.035)^2 = \boxed{\$401.71}$$

b) Monthly $(n=12)$

$$A(2) = 375\left(1+\frac{0.035}{12}\right)^{12\cdot 2}$$

$$= 375(1.0029)^{24} = \boxed{\$402.15}$$

c) Daily $(n=365)$

$$A(2) = 375\left(1+\frac{0.035}{365}\right)^{365\cdot 2}$$

$$= 375(1.000096)^{730} = \boxed{\$402.19}$$

d) Continuously

$$A(t) = A_0 e^{rt} = 375e^{0.035t}$$

$$A(2) = 375e^{0.035\times 2} = \boxed{\$402.19}$$

Section 6.6: 1, 3, 5, 12, 15, 22

1) $\frac{dy}{dx} + y = e^{-x}$

Already in the form
$\frac{dy}{dx} + P(x)\,y = Q(x)$

$\Rightarrow P(x) = 1$

Integrating Factor: $e^{\int p(x)\,dx} = e^{\int 1\,dx} = e^{x}$

$$\frac{dy}{dx}e^{x} + ye^{x} = e^{-x}e^{x} = e^{0} = 1$$

$\Downarrow$ product rule

$$\frac{d}{dx}\left[e^{x}y\right] = 1 \quad \Rightarrow \quad \int \frac{d}{dx}\left[e^{x}y\right]dx = \int 1\,dx$$

$$\Rightarrow \quad e^{x}y = x + C$$

$$y = \frac{x + C}{e^{-x}} = \boxed{xe^{-x} + Ce^{-x}}$$

3) $(1-x^2)\frac{dy}{dx} + xy = ax \quad , \quad |x| < 1$

$$\frac{dy}{dx} + \frac{x}{1-x^2}\,y = \frac{ax}{1-x^2}$$

$\Rightarrow \quad p(x) = \frac{x}{1-x^2} \quad \Rightarrow$ Integrating Factor $= e^{\int \frac{x}{1-x^2}dx}$

note: $\int \frac{x}{1-x^2}\,dx$

$u = 1-x^2$
$du = -2x\,dx$
$dx = \frac{-du}{2x}$

$$= \int \frac{\cancel{x}}{1-x^2}\cdot\left(\frac{-du}{2\cancel{x}}\right)$$

$$= \int \frac{-1}{2} \cdot \frac{1}{u}\, du = -\frac{1}{2} \int \frac{1}{u}\, du = -\frac{1}{2} \ln|u|$$

$$= -\frac{1}{2} \ln|1-x^2| = \frac{-1}{2} \ln(1-x^2) \quad \text{since } |x|<1$$

$$\Rightarrow \text{Integrating Factor} = e^{-\frac{1}{2}\ln(1-x^2)} = e^{\ln\left((1-x^2)^{-1/2}\right)}$$

$$= (1-x^2)^{-1/2}$$

$$\Rightarrow \frac{d}{dx}\left[(1-x^2)^{-1/2} y\right] = \frac{ax(1-x^2)^{-1/2}}{(1-x^2)}$$

$$\Rightarrow \int \frac{d}{dx}\left[(1-x^2)^{-1/2} y\right] dx = \int \frac{ax(1-x^2)^{-1/2}}{(1-x^2)}\, dx$$

$$(1-x^2)^{-1/2} y = \int \frac{ax(1-x^2)^{-1/2}}{(1-x^2)}\, dx$$

we have multiple terms of $(1-x^2)^a$ where a is some number

$$\Rightarrow \text{let} \quad u = (1-x^2)^a$$

$$du = -2x(1-x^2)^{a-1}$$

$\Rightarrow$ we should let $a = 1/2$ so that

$$du = -2x(1-x^2)^{-1/2} \cdot \frac{1}{2}$$

$$\Rightarrow (1-x^2)^{-1/2} y = -1 \int \frac{a \overbrace{\boxed{(-1)x(1-x^2)^{-1/2}\, dx}}^{= du}}{\big(\underbrace{(1-x^2)^{1/2}}_{u}\big)^2} = -a \int \frac{1}{u^2}\, du$$

$$\Rightarrow \quad (1-x^2)^{-1/2} y = -a\int u^{-2}\,du = -a\left[-u^{-1}\right] + C$$

$$\Rightarrow \quad (1-x^2)^{-1/2} y = \frac{a}{u} + C = \frac{a}{(1-x^2)^{1/2}} + C$$

$$\Rightarrow \quad (1-x^2)^{-1/2} y = a(1-x^2)^{-1/2} + C$$

$$\Rightarrow \quad \boxed{y = a + C(1-x^2)^{1/2}}$$

5) $\frac{dy}{dx} - \frac{y}{x} = xe^x$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{1}{x}y = xe^x \qquad p(x) = -\frac{1}{x}$$

$$\Rightarrow \text{Integrating Factor} = e^{\int -\frac{1}{x}dx} = e^{-\int \frac{1}{x}dx} = e^{-\ln(x)}$$

$$e^{-\ln(x)} = e^{\ln(x^{-1})} = x^{-1} = \frac{1}{x}$$

$$\Rightarrow \quad \frac{d}{dx}\left[\frac{1}{x}y\right] = xe^x \cdot \frac{1}{x} = e^x$$

$$\int \frac{d}{dx}\left[\frac{1}{x}y\right]dx = \int e^x$$

$$\frac{1}{x}y = e^x + C$$

$$\boxed{y = xe^x + Cx}$$

12) $y' = e^{2x} - 3y \qquad y=1 \text{ when } x=0$

$$\Rightarrow \quad y' + 3y = e^{2x} \qquad p(x) = 3$$

$$\text{I.F.} = e^{\int 3\,dx} = e^{3x}$$

$$\Rightarrow \quad \frac{d}{dx}\left[e^{3x} y\right] = e^{3x} e^{2x} = e^{5x}$$

$$\int \frac{d}{dx}\left[e^{3x} y\right] dx = \int e^{5x}\, dx$$

$$e^{3x} y = \int e^{5x}\, dx = \frac{1}{5} e^{5x} + C$$

$$y = \frac{\frac{1}{5} e^{5x}}{e^{3x}} + \frac{C}{e^{3x}}$$

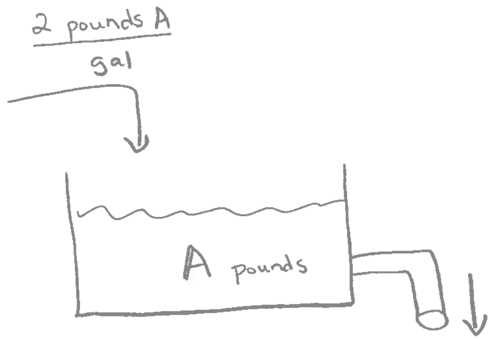
$$y = \frac{1}{5} e^{2x} + Ce^{-3x}$$

$y=1$ when $x=0$ $\Rightarrow$ $1 = \frac{1}{5} e^{0} + Ce^{0}$

$$1 = \frac{1}{5} + C \quad \Rightarrow \quad C = \frac{4}{5}$$

$$\boxed{y = \frac{1}{5} e^{2x} + \frac{4}{5} e^{-3x}}$$

15)



$A_0 = 10$ pounds at $t=0$

$$\frac{dA}{dt} = \underbrace{\frac{2 \text{ pound } A}{\text{gallon}} \cdot \frac{3 \text{ gal}}{\text{min}}}_{\text{In}} - \underbrace{\frac{A \text{ pounds}}{20 \text{ gallons}} \cdot \frac{3 \text{ gal}}{\text{min}}}_{\text{out}}$$

$$\Rightarrow \quad \frac{dA}{dt} = 6 - \frac{3}{20} A$$

$$\frac{dA}{dt} + \frac{3}{20} A = 6 \qquad p(t) = \frac{3}{20}$$

$$\text{Integrating Factor} = e^{\int \frac{3}{20} dt} = e^{\frac{3t}{20}}$$

$$\Rightarrow \quad \frac{d}{dt}\left[e^{\frac{3t}{20}} A\right] = 6e^{\frac{3t}{20}}$$

$$\Rightarrow \quad e^{\frac{3}{20}t} A = \int 6e^{\frac{3}{20}t} dt = 6\int e^{\frac{3}{20}t} dt$$

$$e^{\frac{3}{20}t} A = 6\left[\frac{20}{3} e^{\frac{3}{20}t}\right] + C$$

$$A = 40 + Ce^{-\frac{3}{20}t}$$

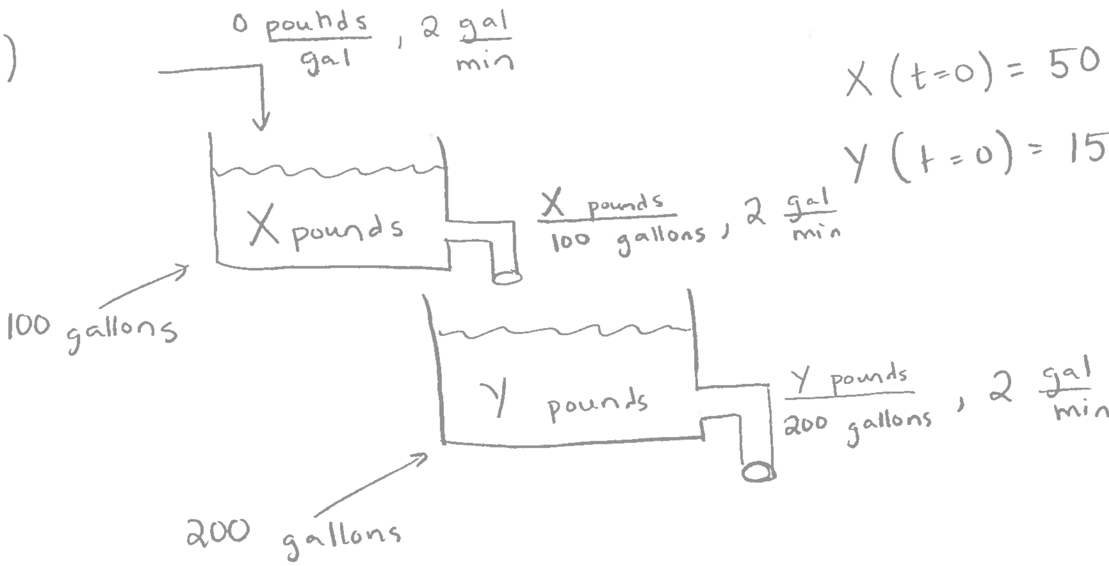
$A = 10$ when $t=0$ $\Rightarrow$ $10 = 40 + Ce^{0} = 40 + C$ $\Rightarrow$ $C = -30$

$$\boxed{A = 40 - 30e^{-\frac{3}{20}t}}$$

$$\Rightarrow \quad A(20) = 40 - 30e^{-\frac{3}{20}(20)}$$

$$= 40 - 30e^{-3} = 40 - \frac{30}{e^3} = \boxed{38.506 \text{ pounds}}$$

22)



$X(t=0) = 50$

$Y(t=0) = 150$

$$\frac{dX}{dt} = \underbrace{\frac{0 \text{ pounds}}{\text{gal}} \cdot \frac{2 \text{ gal}}{\text{min}}}_{\text{IN}} - \underbrace{\frac{X \text{ pounds}}{100 \text{ gal}} \cdot \frac{2 \text{ gal}}{\text{min}}}_{\text{OUT}}$$

$$\frac{dY}{dt} = \underbrace{\frac{X \text{ pounds}}{100 \text{ gal}} \cdot \frac{2 \text{ gal}}{\text{min}}}_{\text{IN}} - \underbrace{\frac{Y \text{ pounds}}{200 \text{ gal}} \cdot \frac{2 \text{ gal}}{\text{min}}}_{\text{OUT}}$$

Since $\frac{dY}{dt}$ depends on X we need to solve for X first

$$\frac{dX}{dt} = -\frac{1}{50}X \qquad \text{seperable}$$

$$\int \frac{dX}{X} = \int -\frac{1}{50}dt \Rightarrow \ln|X| = -\frac{1}{50}t + C$$

$$X = e^{-\frac{1}{50}t}e^{C} = Ce^{-\frac{1}{50}t}$$

$$\Rightarrow \quad X(t=0) = 50 = Ce^{-\frac{1}{50}\cdot 0}$$

$$\Rightarrow \quad C = 50$$

$$\boxed{X = 50e^{-\frac{1}{50}t}}$$

$$\Rightarrow \quad \frac{dY}{dt} = \frac{1}{50}\cdot 50e^{-\frac{1}{50}t} - \frac{1}{100}Y$$

$$\frac{dY}{dt} + \frac{1}{100}Y = e^{-\frac{1}{50}t}$$

First Order Linear, $p(t) = \frac{1}{100}$

$$\text{Integrating Factor} = e^{\int \frac{1}{100}dt} = e^{\frac{1}{100}t}$$

$$\frac{d}{dt}\left[e^{\frac{1}{100}t}Y\right] = e^{-\frac{1}{50}t}e^{\frac{1}{100}t} = e^{-\frac{1}{100}t}$$

$$\Rightarrow \quad e^{\frac{1}{100}t}Y = \int e^{-\frac{1}{100}t}\,dt$$

$$e^{\frac{1}{100}}Y = -100e^{-\frac{1}{100}t} + C$$

$$Y = -100e^{-\frac{1}{50}t} + Ce^{-\frac{1}{100}t}$$

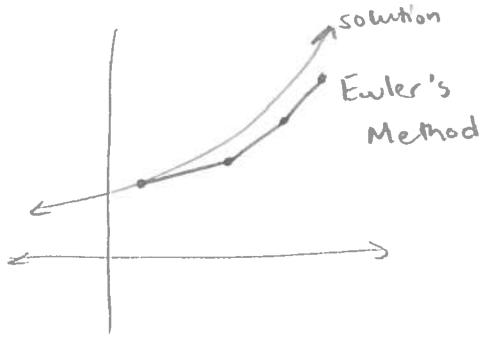
$$Y(t=0) = 150 = -100e^{-\frac{1}{50}(0)} + Ce^{-\frac{1}{100}(0)}$$

$$150 = -100 + C \Rightarrow C = 250$$

$$\boxed{Y(t) = 250e^{-\frac{1}{100}t} - 100e^{-\frac{1}{50}t}}$$

Section 6.7

If the solution of a differential equation is concave up, then Euler's Method will $\boxed{\text{underestimate}}$ the solution



Similarly, if the solution is concave down Euler's Method will $\boxed{\text{overestimate}}$ the solution.