Midterm 3 Practice Midterm Solutions

1 True/False

For each of the following questions respond true if the statement is true and false if the statement is false. If your response is false give a counter example or explain why.

1. It is possible for the sum of infinitely many numbers to converge.

True, for example a Riemann Sum of an integrable function

2. A Riemann sum with a finite partition is exactly equal to the corresponding definite integral.

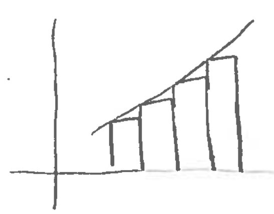
False, only in the limit that $\|P\| \to 0$ or in other words infinitely many rectangles $\Rightarrow$ not finite

3. $\int_a^b f(x)dx = \int_a^c f(x)dx - \int_b^c f(x)dx$.

True, $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Interval Additive Property

4. For an increasing function the left Riemann sum method always overestimates the definite integral.



False, the left Riemann Sum underestimates the definite integral of an increasing function.

2 Free response

1. Evaluate the following sum: $\sum_{k=1}^{7} k\sin(\frac{k\pi}{2})$

$$\sum_{k=1}^{7} k\sin\left(\frac{k\pi}{2}\right) = \underset{k=1}{1\cdot\sin\left(\frac{\pi}{2}\right)} + \underset{k=2}{2\sin(\pi)} + \underset{k=3}{3\sin\left(\frac{3\pi}{2}\right)} + \underset{k=4}{4\sin(2\pi)}$$

$$+ \underset{k=5}{5\sin\left(\frac{5\pi}{2}\right)} + \underset{k=6}{6\sin(3\pi)} + \underset{k=7}{7\sin\left(\frac{7\pi}{2}\right)}$$

$$= 1 + 0 - 3 + 0 + 5 + 0 - 7 = 6 - 10 = \boxed{-4}$$

2. Represent the following sum in Sigma notation and then evaluate it:
S=1+3+5+7+....+15

there are 8 terms being added together $\Rightarrow \sum_{i=1}^{8}$

when $i=1 \Rightarrow 1$

$i=2 \Rightarrow 3$

$i=3 \Rightarrow 5$

$i=4 \Rightarrow 7$

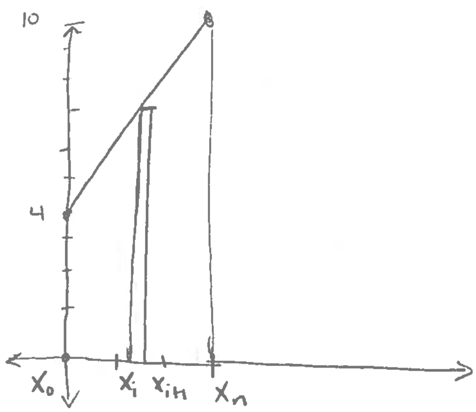
$\Rightarrow$ pattern is $2i-1$

$$\Rightarrow \boxed{S = \sum_{i=1}^{8} 2i-1}$$

$$\Rightarrow S = 2\sum_{i=1}^{8} i - \sum_{i=1}^{8} 1 = \frac{2(8+1)(8)}{2} - 8$$

$$= 72 - 8 = \boxed{64}$$

3. Use the Riemann Sum definition of the definite integral to evaluate
$\int_0^3 (2x+4)dx$



$$\int_0^3 (2x+4)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_i)}_{\text{height}} \underbrace{\Delta x_i}_{\text{width}}$$

$$\text{width} = \frac{\text{total width}}{\text{number rectangles}} = \boxed{\frac{3}{n}}$$

$x_0 = 0$

$x_1 = 0 + \frac{3}{n}$

$x_2 = \frac{3}{n} + \frac{3}{n} = 2\cdot\frac{3}{n}$

$\vdots$

$\boxed{x_i = i\cdot\frac{3}{n}} \implies f(x_i) = 2x_i + 4 = 2\cdot\frac{i\cdot 3}{n} + 4 = \boxed{\frac{6i}{n} + 4}$

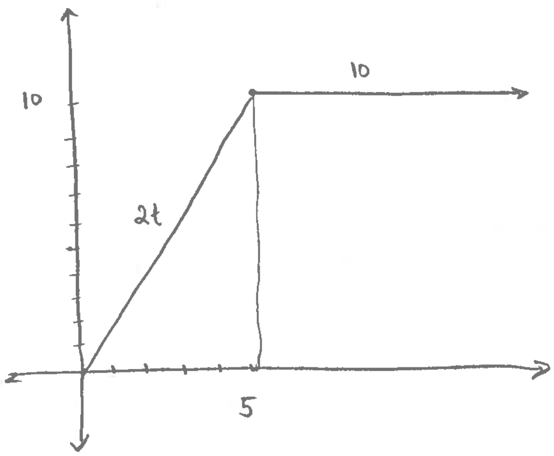
$$\implies \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{6i}{n} + 4\right)\left(\frac{3}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{18i}{n^2} + \frac{12}{n}$$

$$= \lim_{n\to\infty} \frac{18}{n^2} \underbrace{\sum_{i=1}^{n} i}_{\frac{(n+1)(n)}{2}} + \frac{1}{n} \underbrace{\sum_{i=1}^{n} 12}_{12n} = \lim_{n\to\infty} \frac{18(n^2+n)}{n^2\cdot 2} + \frac{12n}{n}$$

$$= \lim_{n\to\infty} 9 + \cancel{\frac{9}{n}}^{\,0} + 12$$

$$= \boxed{21}$$

4. If the velocity function of an object starting at the origin is given as
$v(t) = \begin{cases} 2t & : t \in [0,5] \\ 10 & : t \in [5,\infty) \end{cases}$
Find the location of the object at t=20 and at t=50.



$$d(20) = \int_0^{20} v(t)\,dt$$

$$= \int_0^5 2t\,dt + \int_5^{20} 10\,dt$$

$$= \left[t^2\right]_0^5 + \left[10t\right]_5^{20}$$

$$= (5^2 - 0^2) + (10 \cdot 20 - 10 \cdot 5)$$

$$= 25 + 150 = \boxed{175}$$

$$d(50) = \int_0^{50} v(t)\,dt = \int_0^5 2t\,dt + \int_5^{50} 10\,dt$$

$$= \left[t^2\right]_0^5 + \left[10t\right]_5^{50} = (5^2 - 0) + (10 \cdot 50 - 10 \cdot 5)$$

$$= 25 + 450 = \boxed{475}$$

5. If $G(x) = \int_{x}^{\pi/4} (s-2)\cot(2s)ds$ for $0 < x < \pi/2$ find G'(x)

$$G(x) = \int_{x}^{\pi/4} (s-2)\cot(2s)\,ds \Rightarrow G'(x) = \frac{d}{dx}\int_{x}^{\pi/4} (s-2)\cot(2s)\,ds$$

$$G'(x) = -\underbrace{\frac{d}{dx}\int_{\pi/4}^{x} (s-2)\cot(2s)\,ds}_{(x-2)\cot(2x)} \Rightarrow$$

By the First Fundamental Theorem of Calculus

$$\boxed{G'(x) = -(x-2)\cot(2x)}$$

6. If $H(x) = \int_{-x^2}^{x^2} \frac{t^2}{1+t^2}dt$ find H'(x)

$$H(x) = \int_{-x^2}^{x^2} \frac{t^2}{1+t^2}dt = \int_{-x^2}^{a} \frac{t^2}{1+t^2}dt + \int_{a}^{x^2} \frac{t^2}{1+t^2}dt$$

(Interval Additive Property)

$$= -\int_{a}^{-x^2} \frac{t^2}{1+t^2}dt + \int_{a}^{x^2} \frac{t^2}{1+t^2}dt = -\int_{a}^{v} \frac{t^2}{1+t^2}dt + \int_{a}^{u} \frac{t^2}{1+t^2}$$

let $v = -x^2$ $\quad u = x^2$ $\quad \frac{dv}{dx} = -2x$ $\quad \frac{du}{dx} = 2x$

$$H'(x) = \frac{d}{dx}H(x) = \frac{du}{dx}\frac{d}{du}H(x)$$

First Fundamental Theorem

$$\Rightarrow -\frac{dv}{dx}\frac{d}{dv}\int_{a}^{v}\frac{t^2}{1+t^2}dt + \frac{du}{dx}\frac{d}{du}\int_{a}^{u}\frac{t^2}{1+t^2}dt = -(-2x)\underbrace{\left(\frac{x^4}{1+x^4}\right)}_{f(v)} + 2x\underbrace{\left(\frac{x^4}{1+x^4}\right)}_{f(u)}$$

$$= \boxed{\frac{4x^5}{1+x^4}}$$

7. Evaluate the definite integral $\int_0^2 (2x^4 - 3x^2 + 5)dx$

$$\int_0^2 (2x^4 - 3x^2 + 5)\,dx$$

$$= \left[2 \cdot \frac{1}{5}x^5 - 3 \cdot \frac{1}{3}x^3 + 5x\right]_0^2$$

$$= \left[\frac{2}{5}x^5 - x^3 + 5x\right]_0^2$$

$$= \left(\frac{2}{5}(2)^5 - (2)^3 + 5(2)\right) - \left(\frac{2}{5}(0)^5 - (0)^3 + 5(0)\right)$$

$$= \frac{64}{5} - 8 + 10 = \frac{64}{5} + 2 = \frac{64}{5} + \frac{10}{5}$$

$$= \boxed{\frac{74}{5}}$$

8. Evaluate the definite integral $\int_0^{\sqrt{\pi-4}} \frac{x\sin(\sqrt{x^2+4})}{\sqrt{x^2+4}}dx$

$$\int_0^{\sqrt{\pi-4}} \frac{x \sin\left(\sqrt{x^2+4}\right)}{\sqrt{x^2+4}} \, dx$$

let $u = \sqrt{x^2+4} = (x^2+4)^{1/2}$

$$du = \frac{1}{2}(x^2+4)^{-1/2}(2x)dx = x(x^2+4)^{-1/2}dx = \frac{x\,dx}{\sqrt{x^2+4}}$$

$$\Rightarrow \int_{x=0}^{x=\sqrt{\pi-4}} \sin(u)\,du$$

if $x=0$ then $u = \sqrt{0^2+4} = \sqrt{4} = 2$

if $x = \sqrt{\pi-4}$ then $u = \sqrt{(\sqrt{\pi-4})^2+4}$

$= \sqrt{\pi-4+4} = \sqrt{\pi}$

$$= \int_{u=2}^{u=\sqrt{\pi}} \sin(u)\,du = \Big[-\cos(u)\Big]_2^{\sqrt{\pi}} = -\Big[\cos(u)\Big]_2^{\sqrt{\pi}}$$

$$= -\left(\cos(\sqrt{\pi}) - \cos(2)\right)$$

$$= \boxed{\cos(2) - \cos(\sqrt{\pi})}$$

9. Find the average value of the function $f(x) = x\cos x^2$ on the interval $[0, \sqrt{\pi}]$

$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\sqrt{\pi}-0}\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$$

$u = x^2$

$du = 2x\,dx$

$$\Rightarrow \frac{1}{\sqrt{\pi}}\cdot\frac{1}{2}\int_{x=0}^{x=\sqrt{\pi}} 2x\cos(x^2)\,dx = \frac{1}{2\sqrt{\pi}}\int_{u=0}^{u=\pi}\cos(u)\,du$$

$x = 0 \Rightarrow u = 0^2 = 0$

$x = \sqrt{\pi} \Rightarrow u = (\sqrt{\pi})^2 = \pi$

$$= \frac{1}{2\sqrt{\pi}}\Big[\sin(u)\Big]_0^{\pi} = \frac{1}{2\sqrt{\pi}}\left(\sin(\pi)-\sin(0)\right)$$

$$= \frac{1}{2\sqrt{\pi}}(0-0) = \boxed{0}$$

10. Find all values of c which satisfy the mean value theorem for the function $g(x) = x(1-x)$ on the interval $[0, 1]$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{1-0}\int_0^1 x - x^2\,dx$$

$$= 1\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \left(\frac{1}{2}-\frac{1}{3}\right) - (0-0)$$

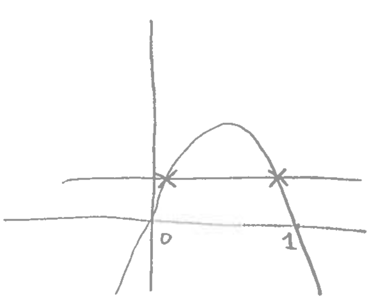
$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\Rightarrow f(c) = \frac{1}{6} = c(1-c) = c - c^2$$

$$\Rightarrow c^2 - c + \frac{1}{6} = 0 \qquad c = \frac{1 \pm \sqrt{1-4(1)(\frac{1}{6})}}{2}$$

8

$$\boxed{c = \frac{1 \pm \sqrt{1/3}}{2} = \frac{3 \pm \sqrt{3}}{6}}$$



Both c's are between 0, 1