Midterm 2 Practice Midterm

1 True/False

For each of the following questions respond true if the statement is true and false if the statement is false. If your response is false give a counter example or explain why.

1. Max/Min values of a function occur at critical points of the function.

   True

2. Max/Min values occur at inflection points of a function.

   False, $f(x) = x^3 \Rightarrow f''(x) = 6x$ has an inflection point at $x=0$ but $x=0$ is not a min/max of f.

3. The First Derivative test can be used for any type of internal critical point.

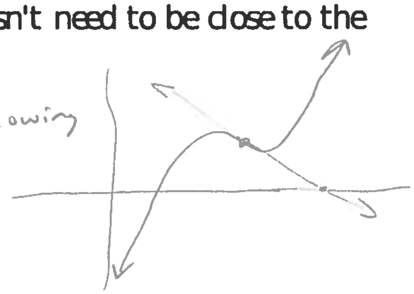
   True

4. The First/Second Derivative test are used to find global max/min values of a function.

   False, the derivative tests are to find Local min/max values

5. The initial guess for Newton's method doesn't need to be close to the root it is trying to find.

   False, consider the following



6. The derivative of the antiderivative of f(x) is equal to f(x).

True

2 Free response

1. Find the linear approximation of the function $f(x) = \sin(4x) + 2x^2$ at the point $x = \frac{\pi}{2}$

slope: $f'(x) = 4\cos(4x) + 4x$

$f'(\frac{\pi}{2}) = 4\cos(2\pi) + 2\pi = 4 + 2\pi$

point: $(\frac{\pi}{2}, f(\frac{\pi}{2})) = (\frac{\pi}{2}, \sin(2\pi) + \frac{\pi^2}{2}) = (\frac{\pi}{2}, \frac{\pi^2}{2})$

point-slope form:

$\Rightarrow \quad y - y_1 = m(x - x_1)$

$$y = (4+2\pi)(x - \frac{\pi}{2}) + \frac{\pi^2}{2}$$

2. Find the global maximum and minimum values of the function $g(x) = 2x^3 + 3x^2 - 4x + 4$ on the interval $I = (-1; 2]$

$g(x) = 2x^3 + 3x^2 - 4x + 4$, $I = (-1, 2]$

critical points: ① Endpoints $\Rightarrow x = 2$

② $f'(x) = 0 \Rightarrow g'(x) = 6x^2 + 6x - 4 = 2(3x^2 + 3x - 2) = 0$

$x = \frac{-3 \pm \sqrt{9 - 4(3)(-2)}}{2(3)} = \frac{-3 \pm \sqrt{33}}{6}$

note: $\sqrt{33} \approx 6 \Rightarrow x \approx \frac{-9}{6}, \frac{3}{6}$ only $x = \frac{1}{6}(\sqrt{33} - 3)$ is in I

③ $f'(x)$ DNE

there are no points where $f'(x)$ is undefined (polynomial)

$\Rightarrow$ global max/min occurs at the critical points

(or there may not be one if $f(-1)$ is above/below)

$$f\left(\frac{-3+\sqrt{33}}{6}\right) \approx f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 4$$

$$= 2\left(\frac{1}{8}\right) + 3\left(\frac{1}{4}\right) - 2 + 4$$

$$= \frac{1}{4} + \frac{3}{4} + 2 = 3$$

$$f(2) = 2(2)^3 + 3(2)^2 - 4(2) + 4$$

$$= 16 + 12 - 8 + 4 = 24$$

$$f(-1) = 2(-1)^3 + 3(-1)^2 - 4(-1) + 4$$

$$= 2(-1) + 3(1) + 4 + 4$$

$$= 9$$

$\Rightarrow$ global max value $= 24$

global min value $= f\left(\frac{-3+\sqrt{33}}{6}\right) \approx 3$

3. Find the regions of increasing/decreasing for the function $h(x) = \frac{x-1}{x}$

$$h(x) = \frac{x-1}{x} \Rightarrow h'(x) = \frac{\frac{d}{dx}[x-1]\,x - (x-1)\frac{d}{dx}[x]}{x^2}$$

$$h'(x) = \frac{x-(x-1)}{x^2} = \frac{1}{x^2} \Rightarrow h'(x) \text{ undefined at } x = 0$$

$h'(x)$: $+$ | $+$ (sign chart, split at $x=0$)

$$h'(1) = \frac{1}{1^2} = 1 > 0$$

$$h'(-1) = \frac{1}{(-1)^2} = 1 > 0$$

$\Rightarrow h(x)$ is increasing for $(-\infty, 0) \cup (0, \infty)$

or $\mathbb{R} - \{0\}$

4. Find the inflection points and regions of concave up/down for the function $f(x) = x^4 + 8x^3 - 2$

$$f(x) = x^4 + 8x^3 - 2$$

$$f'(x) = 4x^3 + 24x^2$$

$$f''(x) = 12x^2 + 48x$$

$$= 12x(x+4)$$

$$f''(x) = 0 = 12x(x+4)$$

$$\Rightarrow x = 0,\ x = -4$$

both single roots

$+$ | $-$ | $+$ (sign chart, split at $x=-4$ and $x=0$)

$$f''(1) = 12(1)(5) = 60 > 0$$

$\Rightarrow$ Inflection points at $x = 0$, $x = -4$

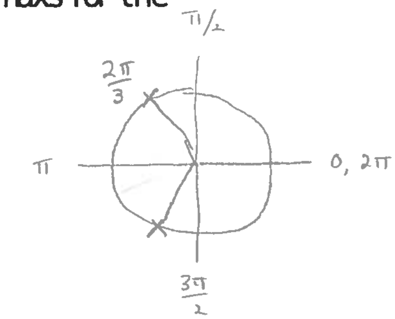
Concave up for $x > 0$, $x < -4$

Concave down for $-4 < x < 0$

For problems 5,6 use an appropriate derivative test:

5. Identify all critical points and determine the local mins/maxs for the function $g(x) = \frac{1}{2}x + \sin(x)$ on the interval $I = (0; 2\pi)$

$g(x) = \frac{1}{2}x + \sin(x)$ $\qquad I = (0, 2\pi)$



$g'(x) = \frac{1}{2} + \cos(x)$

critical points:

① $g'(x) = 0 \Rightarrow \frac{1}{2} + \cos(x) = 0$

$\cos(x) = -\frac{1}{2}$ on $(0, 2\pi)$

$\Rightarrow x = \frac{2\pi}{3}, \quad x = \frac{4\pi}{3}$

② $g'(x)$ undefined nowhere

Can use Second Derivative test

$g''(x) = -\sin(x)$

$g''\left(\frac{2\pi}{3}\right) = -\sin\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2} < 0 \Rightarrow$ local max at $x = \frac{2\pi}{3}$

$g''\left(\frac{4\pi}{3}\right) = -\sin\left(\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{2} > 0 \Rightarrow$ local min at $x = \frac{4\pi}{3}$

6. Identify all critical points and determine the local mins/maxs for the function $h(x) = \frac{x}{x^2+4}$

$h(x) = \frac{x}{x^2+4}$ $\qquad h'(x) = \frac{\frac{d}{dx}[x](x^2+4) - x\frac{d}{dx}[x^2+4]}{(x^2+4)^2}$

$h'(x) = \frac{(x^2+4) - x(2x)}{(x^2+4)^2} = \frac{-x^2+4}{(x^2+4)^2}$

Critical points: ① $h'(x) = 0 \Rightarrow -x^2+4 = 0 \quad x = \pm 2$ both single roots

② $h'(x)$ DNE $\Rightarrow (x^2+4)^2 = 0 \quad x^2+4 > 0$ for all real $x \Rightarrow$ no points

1st D test

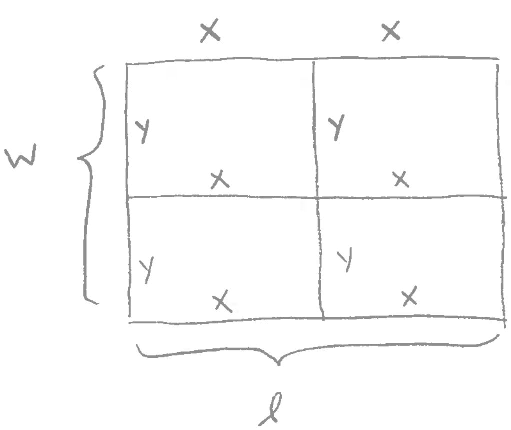
$h'(0) = \frac{4}{16} > 0$

– + –

$x = -2$ local min, $x = 2$ local max

7. Suppose that a farmer has 90 meters of fencing and wants to make 4 identical animal pens by starting with a rectangular pen and subdividing it into equal quarters. What is the maximum ~~volume~~ Area of the 4 pens if the farmer uses up all 90 meters of fencing.

90 meters of fencing, Area = length × width



$$\Rightarrow \quad 6x + 6y = 90$$

$$2x + 2y = 30$$

$$y = \frac{30 - 2x}{2} = 15 - x$$

$$\text{Area} = A = l \times w = (2x)(2y) = 4xy = 4x(15 - x)$$

$$= 60x - 4x^2$$

$$\frac{dA}{dx} = 60 - 8x$$

endpoints: $x > 0$, $6x < 90$

$$\Rightarrow \quad A' = 0 = 60 - 8x$$

$$x = \frac{60}{8} = \frac{15}{2}$$

$\Rightarrow$ min $x = 0$

max $x = 15$

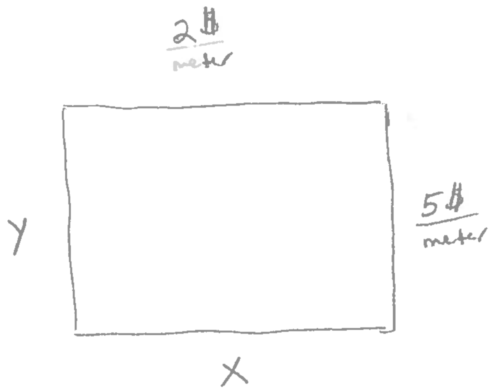
$$\Rightarrow \quad A\left(\frac{15}{2}\right) = 4\left(\frac{15}{2}\right)\left(15 - \frac{15}{2}\right) = 4\left(\frac{15}{2}\right)\left(\frac{15}{2}\right) = 15 \times 15$$

$$= \boxed{225 \text{ meters}^2}$$

$$A(0) = 4(0)(15) = 0$$

$$A(15) = 4(15)(15 - 15) = 0$$

8. A different farmer wants to enclose a single rectangular pen of 400 square meters of area. Due to the winds the east-west sides of the fence have to be built with heavy fencing costing 5 dollars per meter, while the north-south sides can be built of light fencing costing only 2 dollars per meter. What dimensions should the farmer use to minimize the cost of the fence. In other words if we call the east-west fence length w and the north-south fence length l then what are w and l that minimize the cost.



$A = xy = 400 \Rightarrow x = \frac{400}{y}$

total cost = 2x (price x) + 2y (price y)

$\text{Cost} = C = 2x(2) + 2y(5)$

$= 4x + 10y$

$C = 4\left(\frac{400}{y}\right) + 10(y)$

$C = \frac{1600}{y} + 10y = 1600y^{-1} + 10y$

$\frac{dC}{dy} = -\frac{1600}{y^2} + 10 = 0$

$\Rightarrow -\frac{1600}{y^2} = -10$

$-1600 = -10y^2$

$y^2 = 160$

$y = \sqrt{160} = 4\sqrt{10}$

$\Rightarrow x = \frac{400}{4\sqrt{10}} = \frac{100}{\sqrt{10}} = \frac{100\sqrt{10}}{10}$

$= 10\sqrt{10}$

Dimensions:

$x = 10\sqrt{10}$

$y = 4\sqrt{10}$

9a. For the function $f(x) = \frac{3x^5 - 20x^3}{32}$ find the x,y-intercepts

x-intercepts $f(x) = \frac{3x^5 - 20x^3}{32} = 0$

y-intercepts: $f(0) = \boxed{0}$

$\Rightarrow 3x^5 - 20x^3 = 0$

$x^3(3x^2 - 20) = 0$

$3x^2 - 20 = 0$

$3x^2 = 20$

$x^2 = \frac{20}{3} \Rightarrow$

$\boxed{x = 0,\ x = \pm\sqrt{\frac{20}{3}}}$

$\sqrt{\frac{20}{3}} = 2\sqrt{\frac{5}{3}}$ is between 2, 3

9b. Find all critical points of f(x) and where the function is increasing/decreasing

① no endpoints

② $f'(x) = 0$ $\quad f'(x) = \frac{15x^4 - 60x^2}{32} = 0 \Rightarrow 15x^4 - 60x^2 = 0$

$15x^2(x^2 - 4) = 0$

③ $f'(x)$ DNE nowhere (polynomial)

$x = 0$, $x = 2$, $x = -2$

double (x = 0); single (x = 2, x = −2)

Sign chart of f': + | $x=-2$ | − | $x=0$ | − | $x=2$ | +

$f'(1) = 15 - 60 < 0$

$\Rightarrow$ increasing for $x > 2$, $x < -2$

decreasing for $-2 < x < 2$

9c. Find any local minima/maxima

$x = -2$ local max

$x = 2$ local min

1st D test

9d. Find any inflection points of f(x) and where f is concave up/down

$f''(x) = 60x^3 - 120x = 0$

$= 60x(x^2 - 2) = 0$

$x = 0$, $x = -\sqrt{2}$, $x = \sqrt{2}$

single roots

Sign chart of f'': − | $-\sqrt{2}$ | + | 0 | − | $\sqrt{2}$ | +

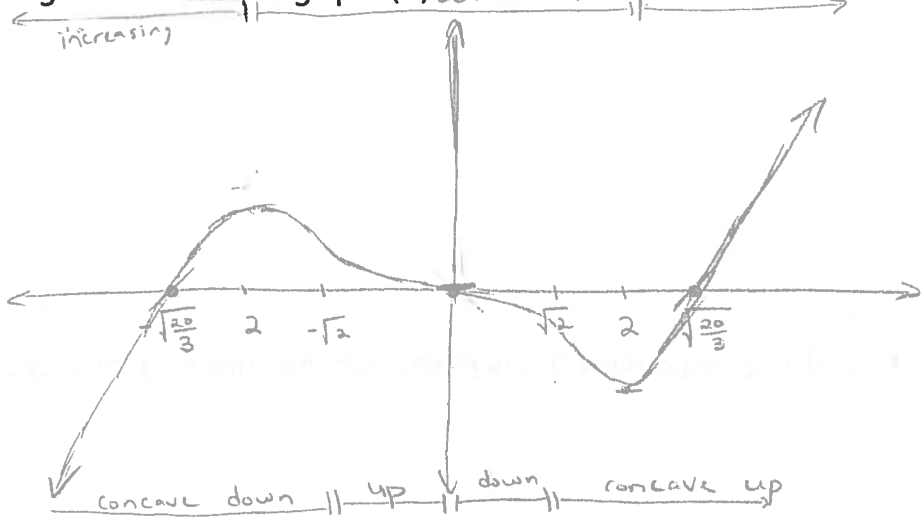
$f''(1) = 60 - 120 < 0$

concave up: $-\sqrt{2} < x < 0$, $x > \sqrt{2}$

concave down: $-\sqrt{2} < x$, $0 < x < \sqrt{2}$

Inflection points: $x = -\sqrt{2}$, $x = 0$, $x = \sqrt{2}$

9e. Using this information graph f(x)



10. Find the antiderivative of $g(x) = 4x^2 + 3x + 2$

$$\int 4x^2 + 3x + 2\,dx = ax^3 + bx^2 + cx + C$$

$$\frac{d}{dx}\left[ax^3 + bx^2 + cx + C\right] = 3ax^2 + 2bx + c = 4x^2 + 3x + 2$$

$\Rightarrow$ $3a = 4$ $\quad a = \frac{4}{3}$

$2b = 3$ $\quad b = \frac{3}{2}$

$c = 2$

$$\Rightarrow \boxed{\frac{4}{3}x^3 + \frac{3}{2}x^2 + 2x + C}$$

11. Find the antiderivative of $h(x) = \sin(x) - \cos(x) + \sec(x)\tan(x)$

$$\int \sin(x) - \cos(x) + \sec(x)\tan(x)\,dx$$

$$= \int \sin(x)\,dx - \int \cos(x)\,dx + \int \sec(x)\tan(x)\,dx$$

$$\int \sin(x) = -\cos(x)$$

$$= \boxed{-\cos(x) - \sin(x) + \sec(x) + C}$$

$$\int \cos(x) = \sin(x)$$

$$\frac{d}{dx}\left[\sec(x)\right] = \sec(x)\tan(x)$$