Midterm 1 Practice Midterm

1 True/False

For each of the following questions respond true if the statement is true and false if the statement is false. If your response is false give a counter example or explain why.

1. Differentiable functions are continuous.

   True, Theorem 2.2 A

2. The derivative is defined at every point in a function's domain.

   False, $f(x) = |x|$ is a function. The derivative is undefined at $x=0$ (jump)

3. The second derivative of sin(x) is equal to the 3rd derivative of cos(x).

   $\frac{d}{dx}[\sin(x)] = \cos(x)$

   $\frac{d^2}{dx^2}[\sin(x)] = \frac{d}{dx}(\cos(x)) = -\sin(x)$

   $\frac{d^3}{dx^3}[\cos(x)] = \frac{d^2}{dx^2}[-\sin(x)] = \sin(x)$

   $\Rightarrow$ False

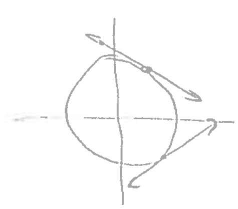
4. If the left and right limits exist at a point the function has a limit at that point.

   True, Theorem 1.1 A

5. The derivative is only defined for functions.

   False, we can find tangent lines for curves which aren't functions with implicit differentiation.

   For example, circles



2 Free response

For the following questions evaluate the limit or state why it does not exist:

1.
$$\lim_{x\to\pi^+} \frac{x^2+4x-\sin(x)}{x-\pi}$$

at $x = \pi$, we have $\frac{\pi^2+4\pi}{0}$ so there is a verticle asymptote at $x=\pi$

for values of $x > \pi$ like $x = 3.5$ we have

$$\frac{(3.5)^2 + 4(3.5) - \sin(3.5)}{3.5 - \pi} \approx \frac{30}{.5} > 0$$ $\Rightarrow$ we are positive and asymptoting up

$$\Rightarrow \lim_{x\to\pi^+} f(x) = \boxed{\infty}$$

2.
$$\lim_{x\to 3} \frac{x-\sin(x-3)-3}{x-3}$$

$$= \lim_{x\to 3} \frac{(x-3)-\sin(x-3)}{x-3} = \lim_{x\to 3}\frac{x-3}{x-3} - \lim_{x\to 3}\frac{\sin(x-3)}{x-3}$$

$$= \lim_{x\to 3} 1 - \lim_{x\to 3}\frac{\sin(x-3)}{x-3}$$

let $y = x-3$, then

$$\lim_{x\to 3}\frac{\sin(x-3)}{x-3} = \lim_{y\to 0}\frac{\sin(y)}{y} = 1$$

$$= 1 - 1 = \boxed{0}$$

3.
$$\lim_{x\to -2^-} x\sqrt{x+2}$$

as $x \to 2^-$ $\sqrt{x+2}$ DNE

$$\Rightarrow \lim_{x\to 2^-} x\sqrt{x+2} \quad \boxed{\text{DNE}}$$

4.

$$\lim_{x\to-\infty}\frac{\sqrt{4-x}}{x+1}$$

$$\lim_{x\to-\infty}\frac{\sqrt{4-x}}{x+1} = \lim_{x\to-\infty}\frac{\frac{1}{x}\sqrt{4-x}}{\frac{x+1}{x}} = \lim_{x\to-\infty}\frac{\sqrt{\frac{4}{x^2}-\frac{x}{x^2}}}{1+\frac{1}{x}}$$

$$= \lim_{x\to-\infty}\frac{\sqrt{4\left(\frac{1}{x^2}\right)-\frac{1}{x}}}{1+\frac{1}{x}} = \frac{\sqrt{4(0)-0}}{1+0} = \frac{0}{1} = \boxed{0}$$

Use the limit definition of the derivative for problems 5,6:

5. Find $f'(x)$ if $f(x) = x^2 + x + 3$

$$\lim_{h\to0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to0}\frac{(x+h)^2+(x+h)+3-(x^2+x+3)}{h}$$

$$= \lim_{h\to0}\frac{\cancel{x^2}+2hx+h^2+\cancel{x}+h+\cancel{3}-\cancel{x^2}-\cancel{x}-\cancel{3}}{h} = \lim_{h\to0}\frac{2hx+h^2+h}{h}$$

$$= \lim_{h\to0} 2x+h+1 = 2x+0+1 = \boxed{2x+1}$$

6. Find $g'(4)$ if $g(x) = \frac{x-4}{x}$

$$f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c} \Rightarrow f'(4) = \lim_{x\to4}\frac{\frac{x-4}{x}-\frac{4-4}{4}}{x-4}$$

$$= \lim_{x\to4}\frac{\frac{x-4}{x}-0}{x-4} = \lim_{x\to4}\frac{\frac{x-4}{x}}{\frac{x-4}{1}} = \lim_{x\to4}\frac{x-4}{x(x-4)}$$

$$= \lim_{x\to4}\frac{1}{x} = \boxed{\frac{1}{4}}$$

Evaluate the following derivatives:

7. find $D_x[f(x)]$ if $f(x) = x^2 \sin(x^2)$

$$D_x\left[x^2 \sin(x^2)\right] = D_x\left[x^2\right]\sin(x^2) + D_x\left[\sin(x^2)\right]x^2$$

$$= 2x\sin(x^2) + x^2 D_x\left[\sin(x^2)\right] = 2x\sin(x^2) + x^2\left(2x\cos(x^2)\right)$$

$$= \boxed{2x^3\cos(x^2) + 2x\sin(x^2)}$$

$$D_x\left[\sin(x^2)\right] = 2x\cos(x^2)$$

$g(x) = x^2$ $\quad g'(x) = 2x$

$f(x) = \sin(x)$ $\quad f'(g(x)) = \cos(x^2)$

8. find $\frac{dg}{dx}$ if $g(x) = \frac{\cos(x)+x^3}{\tan x}$

$$g'(x) = \frac{D_x\left[\cos(x) + x^3\right]\cdot\tan(x) - D_x\left[\tan(x)\right]\cdot\left(\cos(x)+x^3\right)}{\tan^2(x)}$$

$$= \frac{\left(-\sin(x) + x^3\right)\tan(x) - \sec^2(x)\left(\cos(x) + x^3\right)}{\tan^2(x)}$$

$$= \frac{1}{\tan^2(x)}\cdot\left[\frac{-\sin^2(x)}{\cos(x)} + x^3\tan(x) - \frac{1}{\cos(x)} - x^3\sec^2(x)\right]$$

9. Find $h^{(3)}(t)$ if $h(t) = x^4 + 3\cos(x)$

$$h'(t) = D_t\left[x^4 + 3\cos(x)\right] = 0$$

$$h''(t) = 0$$

$$\boxed{h^{(3)}(t) = 0}$$

10. Find $\frac{d^{50}}{dx^{50}}p(x)$ if $p(x) = \sin(x) + \cos(x) + x^{45}$

$$\frac{d^{50}}{dx^{50}}\left[p(x)\right] = \frac{d^2}{dx^2}\left[D_x^{48}\left[p(x)\right]\right] = \frac{d^2}{dx^2}\left[\sin(x) + \cos(x) + 0\right]$$

$$= \frac{d}{dx}\left[\cos(x) - \sin(x)\right] = \boxed{-\sin(x) - \cos(x)}$$

Using implicit differentiation find dy/dx

11. $y = \sqrt{x^2 + \sin(x)}$

$$\frac{dy}{dx} = D_x\left[\left(x^2 + \sin(x)\right)^{1/2}\right]$$

$$g(x) = x^2 + \sin(x) \qquad g'(x) = 2x + \cos(x)$$

$$f(x) = x^{\frac{1}{2}} \qquad f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$f'(g(x)) = \frac{1}{2\sqrt{x^2 + \sin(x)}}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2x + \cos(x)}{2\sqrt{x^2 + \sin(x)}}}$$

12. $x = \sin(y^2) + 2x^3$

$$D_x[x] = D_x\left[\sin(y^2)\right] + 2D_x\left[x^3\right]$$

$$1 = D_x\left[\sin(y^2)\right] + 2(3x^2)$$

$$D_x\left[\sin(y^2)\right] = 1 - 6x^2$$

$$g(y) = y^2 \qquad \frac{d}{dx}[g(y)] = 2y\frac{dy}{dx}$$

$$f(x) = \sin(x) \qquad f'(x) = \cos(x)$$

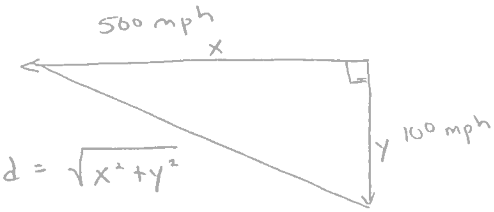
$$f'(g(y)) = \cos(y^2)$$

$$\Rightarrow \quad 2y\cos(y^2)\frac{dy}{dx} = 1 - 6x^2$$

$$\boxed{\frac{dy}{dx} = \frac{1 - 6x^2}{2y\cos(y^2)}}$$

Set up and solve the following related rates problem

13. An airplane is flying west at 500 miles per hour and passes over a train travelling south at 100 miles per hour. How quickly will they be separating from one another after they have travelled for two and a half hours?



$d = \sqrt{x^2+y^2}$

at $2\frac{1}{2}$ hours

$x = 1250$

$y = 250$

$d = \sqrt{x^2+y^2} = (x^2+y^2)^{\frac{1}{2}}$

$D_t[d] = D_t\left[(x^2+y^2)^{\frac{1}{2}}\right]$

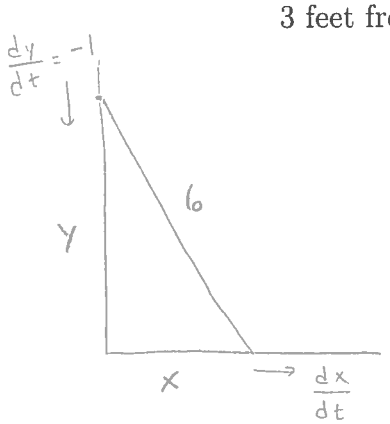
$g(x,y) = x^2+y^2 \qquad \frac{d}{dt}[x^2+y^2] = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$f(x) = x^{\frac{1}{2}} \qquad f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$f'(g) = \frac{1}{2\sqrt{x^2+y^2}}$

$$\Rightarrow D_t[d] = \frac{\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2+y^2}} = \boxed{\frac{1250(500)+250(100)}{\sqrt{(1250)^2+(250)^2}}}$$

14. A 6 foot long ladder is propped up against a wall. It starts to slide down the wall at a speed of 1 foot per second. How quickly is the base of the ladder sliding away from the wall when the top of the ladder is 3 feet from the ground?



30-60-90 triangle

when $y = 3$

$x = \frac{6\cdot\sqrt{3}}{2} = 3\sqrt{3}$

$6 = \sqrt{x^2+y^2}$

$\Rightarrow \frac{d}{dt}[6] = \frac{d}{dt}\left[(x^2+y^2)^{\frac{1}{2}}\right]$

$$\Rightarrow 0 = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2+y^2}}$$

$\Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0$

$\frac{dx}{dt} = \frac{-y}{x}\frac{dy}{dt} = \frac{-3}{3\sqrt{3}}(-1) = \frac{1}{\sqrt{3}}$

$\boxed{\frac{dx}{dt} = \frac{\sqrt{3}}{3}}$