Practice Final

1 True/False

For each of the following questions respond true if the statement is true and false if the statement is false. If your response is false give a counter example or explain why.

1. The limit of a function exists at removable discontinuities.

   True

2. $f(x) = \frac{x}{x-1}$ is continuous on $[-1, 2]$

   False, there is a vertical asymptote at $x = 1$

3. Differentiable functions are continuous

   True

4. The second derivative test can be used for any type of critical point that the first derivative test can be.

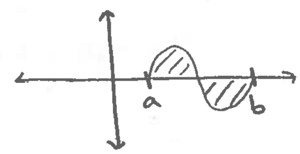
   False, the 2nd derivative test can not be used for f' DNE critical points

5. The derivative of the antiderivative of f(x) is equal to f(x).

   True

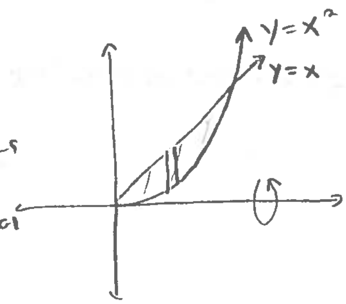
6. If $\int_a^b f(x)dx = 0$ then f(x)=0 for all x between a and b.

False, if area above = area below f(x) does not need to be 0. ex)



7. To find the volume of the region bounded between $y = x^2$ and $y = x$ rotated about the x-axis you should use the method of disks.

False, you should use washers for vertical slices and shells for horizontal slices.



8. In order to find the length of a curve described by an equation that is not a function you must first parametrize the equation.

True.

2 Free response

1a. Find

$$\lim_{x\to -2} \frac{x^2+5x+6}{x-2} = \lim_{x\to -2} \frac{(x+3)(x+2)}{(x-2)}$$

when $x = -2$ there is no division by zero

$\Rightarrow$ rational function is continuous

$$\Rightarrow \lim_{x\to -2} \frac{(x+3)(x+2)}{x-2} = \frac{(-2+3)(-2+2)}{(-2-2)} = \frac{1\cdot 0}{-4}$$

$$= \boxed{0}$$

1b. Find

$$\lim_{x\to 3} \frac{x^2+4x+4}{x-3} = \lim_{x\to 3} \frac{(x+2)(x+2)}{(x-3)}$$

at $x = 3$ we divide by 0, 3 is not also a root of the top $\Rightarrow$ we have a vertical asymptote at $x = 3$

$$\Rightarrow \lim_{x\to 3} \frac{x^2+4x+4}{x-3} \quad \boxed{\text{DNE}}$$

2. Using the limit definition of the derivative find f'(x) if $f(x) = 2x$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2(x+h) - 2x}{h}$$

$$= \lim_{h \to 0} \frac{2x + 2h - 2x}{h} = \lim_{h \to 0} \frac{2h}{h} = \lim_{h \to 0} 2 = \boxed{2}$$

3a. $D_x[x^2 \cos(x)]$

Product Rule $\quad D_x\left[x^2 \cos(x)\right] = D_x\left[x^2\right] \cos(x) + x^2 D_x\left[\cos(x)\right]$

$$= 2x \cos(x) + x^2(-\sin(x))$$

$$= \boxed{2x \cos(x) - x^2 \sin(x)}$$

3b. $\frac{d}{dx}[\sin(\sqrt{(x)})] = \frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

Chain Rule $\quad$ inside $= \sqrt{x} = g(x)$

outside $= \sin(g(x)) = f(g(x))$

$$\Rightarrow \quad \frac{d}{dx}\left[\sin(\sqrt{x})\right] = \cos(\sqrt{x}) \cdot \frac{1}{2} x^{-1/2} = \boxed{\frac{\cos(\sqrt{x})}{2\sqrt{x}}}$$

3c. $D_x[\frac{x+1}{x}]$

Quotient Rule : $D_x\left[\frac{x+1}{x}\right] = \frac{D_x[x+1]\,x - (x+1)\,D_x[x]}{(x)^2}$

$$= \frac{x-(x+1)}{x^2} = \boxed{\frac{-1}{x^2}}$$

4. Using implicit differentiation solve for $\frac{dy}{dx}$ if $y^2 + y\sin(x) = 6x$

$D_x\left[y^2 + y\sin(x)\right] = D_x\left[6x\right]$

product rule

$$\Rightarrow \quad 2y\frac{dy}{dx} + D_x[y]\sin(x) + y\,D_x[\sin(x)] = 6$$

$$\Rightarrow \quad 2y\frac{dy}{dx} + \frac{dy}{dx}\sin(x) + y\cos(x) = 6$$

$$\frac{dy}{dx}\left(2y + \sin(x)\right) = 6 - y\cos(x)$$

$$\boxed{\frac{dy}{dx} = \frac{6 - y\cos(x)}{2y + \sin(x)}}$$

5. Using the derivative test of your choice find all local maxima and minima of $g(x)$ if $g'(x) = \frac{(x+2)^3(x-3)}{(x+2)^2}$

Critical Points : ① $g'(x) = 0 = \dfrac{(x+2)^3(x-3)}{(x+2)^2}$

$\Rightarrow x = 3.$

② $g'(x)$ DNE we have division by 0 when $x = -2$ $\quad \frac{0}{0}$ case

$\Rightarrow x = -2$

$g'(4) = \dfrac{6^3 \cdot 1}{6^2} > 0$

$g'(0) = \dfrac{2^3 \cdot -3}{2^2} < 0$

$g'(-3) = \dfrac{(-1)^3 \cdot (-4)}{(-1)^2} > 0$

1st D test

$+$ $\quad -2 \quad$ $-$ $\quad 3 \quad$ $+$

$\Rightarrow$ $\boxed{\begin{array}{l} x = 3 \text{ local min} \\ x = -2 \text{ local max} \end{array}}$

6. Find all regions of concave up/down if $h(x) = x^4 + 8x^3 - 18x^2 + 4x + 2$

$$h'(x) = 4x^3 + 24x^2 - 36x + 4$$

$$h''(x) = 12x^2 + 48x - 36$$

$\boxed{\begin{array}{ll} \text{Concave up for} & x < -2-\sqrt{7} \\ & x > -2+\sqrt{7} \\ \text{Concave down for} & -2-\sqrt{7} < x < -2+\sqrt{7} \end{array}}$

find $h''(x) = 0$ $\Rightarrow$ $12(x^2 + 4x - 3) = 0$

$$x^2 + 4x - 3 = 0$$

$$x = \frac{-4 \pm \sqrt{16 - 4(1)(-3)}}{2} = \frac{-4 \pm \sqrt{28}}{2} = \frac{-4 \pm 2\sqrt{7}}{2}$$

$$= -2 \pm \sqrt{7}$$

note $\sqrt{7}$ is between 2 and 3

$h''(1) = 12 + 48 - 36 > 0$

$h''(-2) = 12(4) - 48(2) - 36 < 0$

$+$ $\quad -2-\sqrt{7} \quad$ $-$ $\quad -2+\sqrt{7} \quad$ $+$

7. Find the antiderivative of $x^3 - 2x + 4\sin x$

$$\int x^3 - 2x + 4\sin(x)$$

$$= \boxed{\frac{1}{4}x^4 - x^2 - 4\cos(x) + C}$$

Check: $D_x\left[\frac{1}{4}x^4 - x^2 - 4\cos(x) + C\right]$

$$= \frac{1}{4}4x^3 - 2x - (-4\sin(x))$$

$$= x^3 - 2x + 4\sin(x) \checkmark$$

8. If $f(x) = \int_{x^2}^{3} (3t+7)dt$ find $f'(x)$

$$f(x) = -\int_3^{x^2} 3t + 7 \; dt$$

want to use first fundamental theorem of calculus

$$f'(x) = -\frac{d}{dx}\int_3^{x^2} 3t + 7 \; dt$$

let $u = x^2$ $\quad \frac{du}{dx} = 2x$

$$= -\frac{d}{dx}\int_3^{u} (3t+7)\,dt$$

$$\frac{d}{dx} = \frac{du}{dx} \cdot \frac{d}{du}$$

$$= -\left(\frac{du}{dx}\right)\underbrace{\frac{d}{du}\int_3^{u}(3t+7)\,dt}_{\text{FFTC}} = -(2x)(3u+7)$$

$$= -2x(3x^2+7)$$

$$= \boxed{-6x^3 - 14x}$$

should be $\sqrt{\pi}$

9. Evaluate $\int_0^{\sqrt{x}} 4x \sin(x^2)dx$

let $x^2 = u$

$du = 2x\,dx$

$\Rightarrow \int_{x=0}^{x=\sqrt{\pi}} 2\sin(u)\,du$

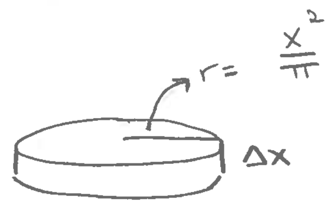
$= 2\int_{u=0}^{u=\pi} \sin(u)\,du = 2\left[-\cos(u)\right]_0^{\pi}$

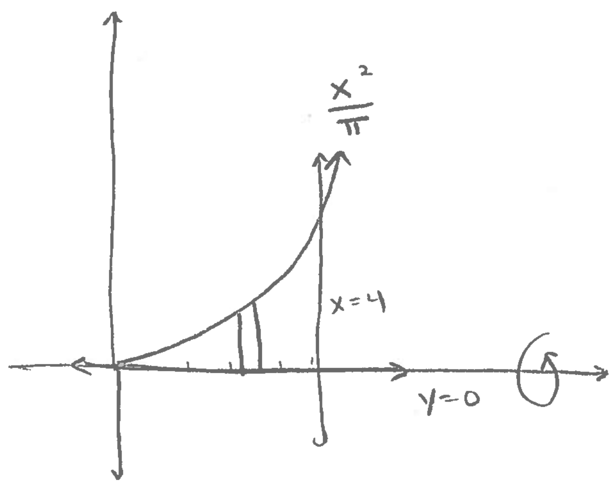
$= -2\left(\cos(\pi) - \cos(0)\right)$

$= -2(-1-1) = -2 \cdot -2$

$= \boxed{4}$

10. Find the volume of the resulting solid of rotation when the region bounded by the curves $y = \frac{x^2}{\pi}$, x=4, y=0 is rotated about the x-axis.





$$\Delta V = \pi r^2 h$$

$$= \pi \left(\frac{x^2}{\pi}\right)^2 \Delta x$$

$$\Rightarrow \int_{x=0}^{x=4} \pi \left(\frac{x^2}{\pi}\right)^2 dx = \frac{1}{\pi} \int_0^4 x^4 \, dx$$

$$= \frac{1}{\pi} \left[\frac{x^5}{5} \right]_0^4 = \frac{1}{5\pi} \left(4^5 - 0^5 \right)$$

$$= \frac{4^5}{5\pi} = \boxed{\frac{1024}{5\pi}}$$

11. Find the length of the curve of $y = \frac{2}{3}x^{3/2}$ between x=0 and x=4

y is a function of x $\Rightarrow$ no need to parametrize

$$\Rightarrow \quad L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{2}{3}x^{3/2}\right] = x^{1/2} = \sqrt{x}$$

$$\Rightarrow \quad L = \int_{x=0}^{x=4} \sqrt{1 + (\sqrt{x})^2}\, dx = \int_{x=0}^{x=4} \sqrt{1+x}\, dx$$

Let $u = 1 + x$

$du = dx$

$$= \int_{u=1}^{u=5} \sqrt{u}\, du$$

$x = 0 \Rightarrow u = 1$

$x = 4 \Rightarrow u = 5$

$$= \left[\frac{2}{3}u^{3/2}\right]_1^5 = \frac{2}{3}\left(5^{3/2} - 1^{3/2}\right)$$

$$= \boxed{\frac{2}{3}\left(\sqrt{125} - 1\right)}$$