HW 6 Solutions

3.3: 2, 7, 9, 16, 17, 24, 29, 32

3.4: 10, 11, 19, 32, 34, 43

Chapter 3 Section 3

2) $f(x) = x^3 - 12x + \pi$

Critical Points: 1 $f'(x) = 0$

2 $f'(x)$ DNE $\rightarrow$ none, f is a polynomial

$f'(x) = 3x^2 - 12 = 0$

$3(x^2 - 4) = 0$

$3(x+2)(x-2) = 0$

$x = 2, \quad x = -2$ both single roots

$f'(-3) = 3(-1)(-5) = 15 > 0$

$f'(0) = 3(2)(-2) = -12 < 0$

$f'(3) = 3(5)(1) = 15 > 0$

By the 1st Derivative test $x = -2$ is a local maximum

$x = 2$ is a local minimum

2nd D test: $f''(x) = 6x$

$f''(-2) = -12 < 0 \Rightarrow x = -2$ local max

$f''(2) = 12 > 0 \Rightarrow x = 2$ local min

7) $f(x) = \dfrac{x}{x^2+4}$

critical points: ① $f'(x) = 0$

$$f'(x) = \frac{\frac{d}{dx}[x](x^2+4) - \frac{d}{dx}[x^2+4]\,x}{(x^2+4)^2}$$

$$= \frac{x^2+4-(2x)(x)}{(x^2+4)^2} = \frac{-x^2+4}{(x^2+4)^2} = 0$$

$$\Rightarrow -x^2+4 = 0 \qquad \boxed{x = \pm 2}$$

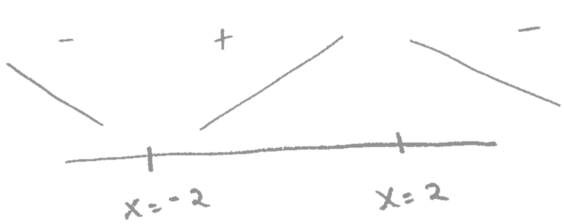
both single roots

② $f'(x)$ DNE

$$\Rightarrow (x^2+4)^2 = 0$$

$$x^2+4 = 0$$

There are no real x for which this is true



$$f'(-3) = \frac{-(-3)^2+4}{(-3^2+4)^2} = \frac{-9+4}{13^2} = \frac{-5}{13^2} < 0$$

$\Rightarrow$ local min at $x = -2$ by 1st D test
local max at $x = 2$

2nd D-test:
$$f''(x) = \frac{\frac{d}{dx}[-x^2+4](x^2+4)^2 - (-x^2+4)\frac{d}{dx}\left[(x^2+4)^2\right]}{(x^2+4)^4}$$

$$= \frac{(-2x)(x^2+4)^2 - (-x^2+4)\left(2(x^2+4)(2x)\right)}{(x^2+4)^4}$$

$$= \frac{-2x(x^2+4)\left(x^2+4+(-x^2+4)(2)\right)}{(x^2+4)^4} = \frac{-2x(-x^2+12)}{(x^2+4)^3}$$

$$f''(2) = \frac{-4(-4+12)}{8^3} < 0 \Rightarrow \text{local max at } x=2$$

$$f''(-2) = \frac{4(-4+12)}{8^3} > 0 \Rightarrow \text{local min at } x=-2$$

9) $h(y) = y^2 - \frac{1}{y} = y^2 - y^{-1}$

Critical points: ① $h'(y) = 0$

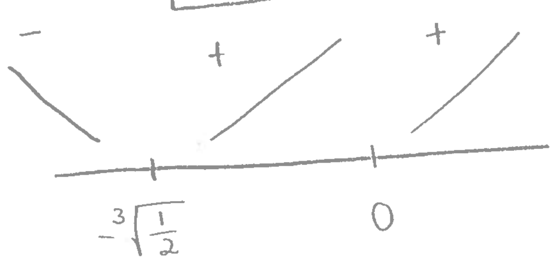
$$h'(y) = 2y + \frac{1}{y^2} = 0$$

$$2y = \frac{-1}{y^2}$$

$$2y^3 = -1 \Rightarrow y^3 = \frac{-1}{2} \Rightarrow \boxed{y = \sqrt[3]{\frac{-1}{2}}}$$

② $h'(y)$ DNE

$\Rightarrow \boxed{y = 0}$ division by 0



$$h'(1) = 2(1) + \frac{1}{1} = 3 > 0$$

$$h'(-2) = 2(-2) + \frac{1}{-2^2} = -4 + \frac{1}{4} < 0$$

$$h'\left(-\frac{.1}{100}\right) = 2\left(-\frac{1}{100}\right) + \frac{1}{(-1/100)^2}$$

$$= -\frac{2}{100} + 10000 > 0$$

$\Rightarrow y = \sqrt[3]{\frac{-1}{2}}$ is a local min

2nd D-test $h''(y) = 2 - 2y^{-3} = 2 - \frac{2}{y^3}$

$$h''\left(\sqrt[3]{\frac{-1}{2}}\right) = 2 - \frac{2}{\left(\sqrt[3]{\frac{-1}{2}}\right)^3} = 2 - \frac{2}{-1/2} = 2 + 4 = 6 > 0$$

$\Rightarrow$ local min

16) $r(s) = 3s + s^{2/5}$

Critical points: ① $r'(s) = 0$

$$r'(s) = 3 + \frac{2}{5}s^{-3/5} = 3 + \frac{2}{5\sqrt[5]{s^3}} = 0$$

$$3 = \frac{-2}{5\sqrt[5]{s^3}} \Rightarrow 15\sqrt[5]{s^3} = -2$$

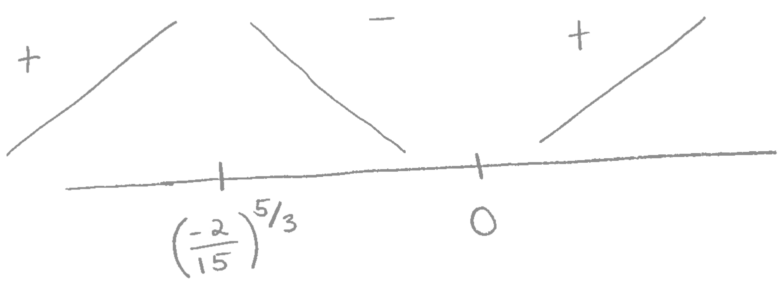
$$\sqrt[5]{s^3} = \frac{-2}{15}$$

$$s^3 = \left(\frac{-2}{15}\right)^5$$

$$\boxed{s = \left(\frac{-2}{15}\right)^{5/3}}$$

(4)

(2) $r'(s)$ DNE when $s=0$



$$r'(1) = 3 + \frac{2}{5}(1)^{-3/5} = 3 + \frac{2}{5} > 0$$

$$r'(-100) = 3 + \frac{2}{5}(-100)^{-3/5}$$

$$= 3 + \frac{2}{5\sqrt[5]{(-100)^3}}$$

$$= 3 + \frac{2}{5\sqrt[5]{-10^6}} = 3 + \frac{2}{50\sqrt[5]{-10}} > 0$$

$$r'\left(-\frac{1}{100}\right) = 3 + \frac{2}{5\sqrt[5]{\left(\frac{-1}{100}\right)^3}}$$

$$= 3 + \frac{2}{5\left(\frac{-1}{10^6}\right)^{1/5}} = 3 + \frac{2}{-5 \cdot \frac{1}{10}\left(\frac{1}{10}\right)^{1/5}}$$

$$= 3 - \frac{20\sqrt[5]{10}}{5}$$

$$= 3 - 4\sqrt[5]{10} < 0$$

$\Rightarrow$ $s = \left(\frac{-2}{15}\right)^{5/3}$ is a local max

$s = 0$ is a local min

17) $f(t) = t - \frac{1}{t}$, $t \neq 0$

(1) $f'(t) = 1 + t^{-2}$

$$= 1 + \frac{1}{t^2} = 0$$

$1 = \frac{-1}{t^2}$ $\quad t^2 = -1$ $\Rightarrow$ no real roots

$f'(t)$ is only undefined at $t = 0$ which is not in the domain $\Rightarrow$ no critical points

$\Rightarrow$ no local min/max

24) $h(x) = \dfrac{1}{x^2+4}$ on $[0, \infty)$

critical points : ① $\boxed{x = 0 \quad \text{also } x \to \infty}$ Endpoints we should check

② $h'(x) = 0$

$$h'(x) = \frac{d}{dx}\left[(x^2+4)^{-1}\right] \qquad \text{chain rule}$$

$$= -1(x^2+4)^{-2}(2x)$$

$$= \frac{-2x}{(x^2+4)^2} = 0 \quad \Rightarrow \quad \boxed{x = 0}$$

③ $h'(x)$ DNE

$x^2+4 > 0$ for all x ⇒ no points

$$h(0) = \frac{1}{0^2+4} = \frac{1}{4} \qquad \text{global max} \quad x = 0$$

$$\lim_{x \to \infty} h(x) = \lim_{x \to \infty} \frac{1}{x^2+4} = 0 \qquad \text{(not included ⇒ not a min)}$$

There is no global min

29) $H(x) = |x^2 - 1|$ on $[-2, 2]$

① endpoints $\boxed{x = -2, \quad x = 2}$

② $H'(x) = 0$ Note $H(x) = \begin{cases} x^2 - 1 & [-2,-1] \cup [1,2] \\ 1 - x^2 & [-1,1] \end{cases}$

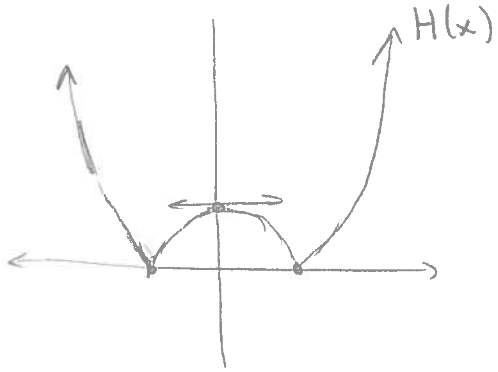
$$\Rightarrow H'(x) = \begin{cases} 2x & x \le -1, \quad x \ge 1 \\ -2x & -1 \le x \le 1 \end{cases}$$

⇒ $H'(x) = 0$ for $\boxed{x = 0}$

$H'(x)$ DNE

this will happen at corners, where the sign of the derivative changes at a point where $H' \neq 0$

this happens for $x = -1$, $x = 1$ (corners)



$$\Rightarrow \quad H(-2) = |(-2)^2 - 1| = |4-1| = 3$$

$$H(-1) = |(-1)^2 - 1| = |1-1| = 0$$

$$H(0) = |0^2 - 1| = |-1| = 1$$

$$H(1) = |1^2 - 1| = |0| = 0$$

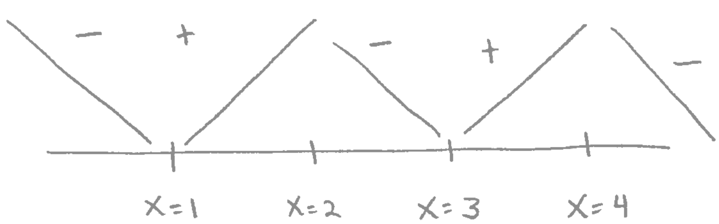
$$H(2) = |2^2 - 1| = |4-1| = 3$$

$\Rightarrow$ Global Max $= 3$ at $x = \pm 2$

Global Min $= 0$ at $x = \pm 1$

32) $f'(x) = -(x-1)(x-2)(x-3)(x-4)$

note all roots are single roots $\Rightarrow$ sign changes



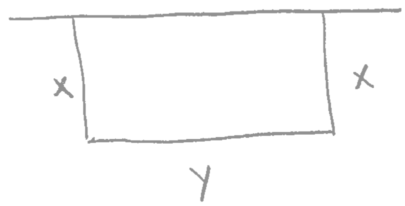
$$f'(0) = -(-1)(-2)(-3)(-4)$$
$$= -24 < 0$$

$\Rightarrow$ $x=1$, $x=3$ are local mins

$x=2$, $x=4$ are local maxs

Chapter 3 Section 4:

10)



80 feet of fence

want maximum area

Area $= xy$

note: total fence $= 2x + y = 80$

$$y = 80 - 2x$$
$$y = 2(40 - x)$$

$\Rightarrow$ Area $= x(2(40-x)) = 2x(40-x)$

$$A = 80x - 2x^2$$

endpoints:

min $x = 0$

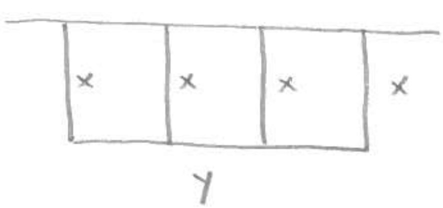
max $x = 40$

$$\frac{dA}{dx} = 80 - 4x = 0$$

$x = 20 \Rightarrow y = 40$

Area $= (20)(40) = \boxed{800 \text{ ft}^2}$

11)



80 ft fence $= 4x + y$

Area $= xy$

$A = xy$

note: $4x + y = 80$

$$y = 80 - 4x = 4(20 - x)$$

$\Rightarrow A = x(4(20-x)) = 4x(20-x) = 80x - 4x^2$

min $x = 0$

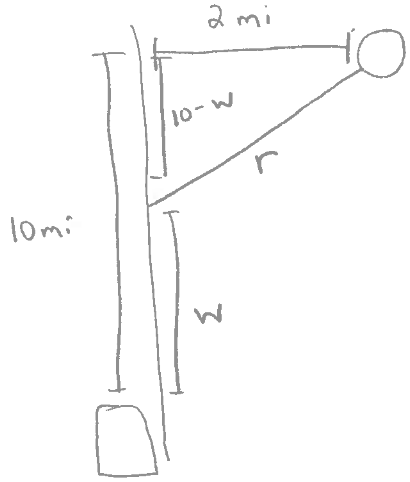
max $x = 20$

$$\frac{dA}{dx} = 80 - 8x = 0$$

$x = 10 \Rightarrow y = 40$

Area $= (10)(40) = \boxed{400 \text{ ft}^2}$

19)



row speed = 3 mph

walk speed = 4 mph

row distance = r

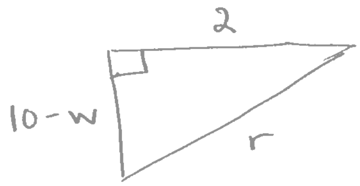
walk distance = w

velocity = $\frac{\text{distance}}{\text{time}}$

time = $\frac{\text{distance}}{\text{velocity}}$

$$\text{Time} = \frac{r}{3} + \frac{w}{4}$$

need a relationship between r + w, use the triangle



$$(10-w)^2 + (2)^2 = r^2$$

$$r = \sqrt{(10-w)^2 + 4}$$

$$\text{Time} = T = \frac{1}{3}\sqrt{(10-w)^2+4} + \frac{1}{4}w$$

w min = 0

w max = 10

$$= \frac{1}{3}\left(100 - 20w + w^2 + 4\right)^{1/2} + \frac{1}{4}w$$

$$= \frac{1}{3}\left(104 - 20w + w^2\right)^{1/2} + \frac{1}{4}w$$

$$\frac{dT}{dw} = \frac{1}{3}\cdot\frac{1}{2}\left(104 - 20w + w^2\right)^{-1/2}(2w-20) + \frac{1}{4} = 0$$

$$\Rightarrow \frac{1}{6}\,\frac{2(w-10)}{(w^2-20w+104)^{1/2}} + \frac{1}{4} = 0$$

$$\frac{w-10}{3(w^2-20w+104)^{1/2}} = -\frac{1}{4}$$

$$4(w-10) = -3\left(w^2-20w+104\right)^{1/2}$$

$$\left(w^2-20w+104\right)^{1/2} = -\frac{4}{3}(w-10)$$

square both sides

$$w^2 - 20w + 104 = \frac{16}{9}(w-10)^2$$

$$w^2 - 20w + 104 = \frac{16}{9}(w^2 - 20w + 100)$$

$$\Rightarrow \frac{7}{9}w^2 - \frac{140}{9}w + \frac{664}{9} = 0$$

solve with quadratic formula and calculator to get

$\boxed{w = 10 - \frac{6}{7}\sqrt{7}}$ → only this one is between $(0,10)$

$w = 10 + \frac{6}{7}\sqrt{7}$

plugging in we get

$$\text{Time} = T\left(10 - \frac{6}{7}\sqrt{7}\right) = \frac{1}{6}\left(15 + \sqrt{7}\right) \text{ hours}$$

If $w = 10 - \frac{6\sqrt{7}}{7}$ then she should land

$10 - \left(10 - \frac{6\sqrt{7}}{7}\right) = \frac{6\sqrt{7}}{7} = \frac{6}{\sqrt{7}}$ miles downshore from the island

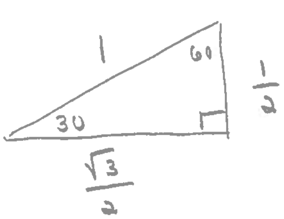
32) $x = \sin(2t) + \sqrt{3}\cos(2t)$

farthest away when $v = 0 = \frac{dx}{dt}$

$$\Rightarrow \frac{dx}{dt} = 2\cos(2t) - 2\sqrt{3}\sin(2t) = 0$$

$$\Rightarrow \cos(2t) = \sqrt{3}\sin(2t)$$

this occurs for 30-60-90 triangles

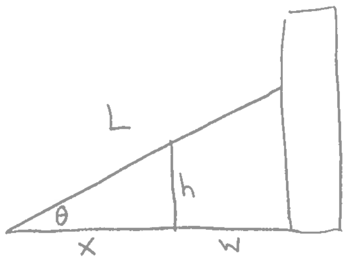


$\Rightarrow 2t = 30 \text{ degrees} = \frac{\pi}{6}$

$t = \frac{\pi}{12}$ $\Rightarrow x\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{6}\right) + \sqrt{3}\cos\left(\frac{\pi}{6}\right)$

$= \frac{1}{2} + \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = \boxed{2}$

34)



$\cos = \frac{\text{adjacent}}{\text{hypotenus}}$

$\sin = \frac{\text{opposite}}{\text{hypotenus}}$

$\tan = \frac{\text{opposite}}{\text{adjacent}}$

$\Rightarrow \quad \cos(\theta) = \frac{x + w}{L}$ and $\tan(\theta) = \frac{h}{x}$

we want to solve for L (minimum)

$L = \frac{x + w}{\cos\theta} \qquad x = \frac{h}{\tan\theta}$

$$\Rightarrow L = \frac{\left(\frac{h}{\tan\theta}\right) + w}{\cos\theta} = \frac{\frac{h\cos\theta}{\sin\theta} + w}{\cos\theta} = \frac{\frac{h\cos\theta + w\sin\theta}{\sin\theta}}{\frac{\cos\theta}{1}}$$

$$= \frac{h\cos\theta + w\sin\theta}{\sin\theta\cos\theta} = \frac{h}{\sin\theta} + \frac{w}{\cos\theta}$$

$$\Rightarrow \frac{dL}{d\theta} = \frac{\frac{d}{d\theta}[h]\sin\theta - h\frac{d}{d\theta}[\sin\theta]}{\sin^2\theta} + \frac{\frac{d}{d\theta}[w]\cos\theta - \frac{d}{d\theta}[\cos\theta]w}{\cos^2\theta}$$

h, w independent of θ

$\Rightarrow \frac{dh}{d\theta} = 0, \quad \frac{dw}{d\theta} = 0$

$$\Rightarrow \frac{dL}{d\theta} = \frac{-h\cos\theta}{\sin^2\theta} + \frac{-(-\sin\theta)w}{\cos^2\theta} = 0$$

$$\Rightarrow \frac{w\sin\theta}{\cos^2\theta} = \frac{h\cos\theta}{\sin^2\theta}$$

$$w\sin^3\theta = h\cos^3\theta$$

$$\frac{\sin^3\theta}{\cos^3\theta} = \frac{h}{w}$$

$\sin\theta = \sqrt[3]{\frac{h}{w}}\cos\theta$

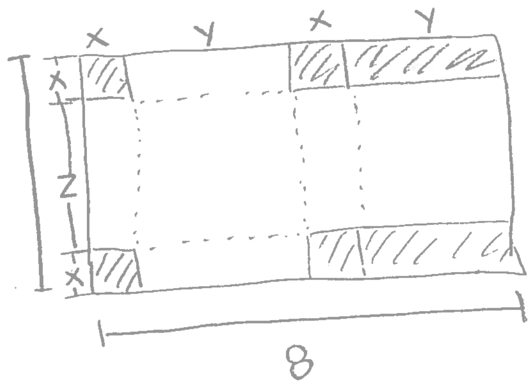
$\cos\theta = \sqrt[3]{\frac{w}{h}}\sin\theta$

$\theta = \tan^{-1}\left(\sqrt[3]{\frac{h}{w}}\right)$

$$\Rightarrow \quad L = \frac{h}{\sin\theta} + \frac{w}{\cos\theta} \qquad \text{where} \qquad w\sin^3\theta = h\cos^3\theta$$

we can't solve completely for L without being given h, w

43)



covered box

note: $2x + 2y = 8$

$2x + z = 5$

$$\text{Volume} = l \times w \times h = z(x)(y) = V$$

we need V for just 1 variable, lets choose x.

$$\Rightarrow \quad 2x + z = 5$$
$$z = 5 - 2x$$

$$2x + 2y = 8$$
$$2y = 8 - 2x$$
$$y = 4 - x$$

min $x = 0$

max $x = \frac{5}{2}$

$$\Rightarrow \quad V = x(4-x)(5-2x)$$
$$= (4x - x^2)(5-2x)$$
$$= 20x - 5x^2 - 8x^2 + 2x^3$$
$$= 2x^3 - 13x^2 + 20x$$

$$\frac{dV}{dx} = 6x^2 - 26x + 20 = 0$$
$$2(3x^2 - 13x + 10) = 0$$
$$2(x-1)(3x-10) = 0$$
$$x = 1, \quad x = \frac{10}{3}$$

$$V'' = 12x - 26$$

$\Rightarrow$ $V''(1) = 12 - 26 = -14 < 0$ $\Rightarrow$ local max

$V''\left(\frac{10}{3}\right) = 12\left(\frac{10}{3}\right) - 26 = 40 - 26 = 14 > 0$ $\Rightarrow$ local min

$\Rightarrow$ maximum volume occurs for $x = 1$

$\Rightarrow$ $Z = 5 - 2(1) = 3$, $y = 4 - (1) = 3$

$$\boxed{V = 1 \times 3 \times 3 = 9}$$