Homework 3 Solutions

Section 1.6: 1, 4, 10, 20, 21, 24, 32

1) $f(x) = (x-3)(x+4) = x^2 - 3x + 4x - 12$

$= x^2 + x - 12$ is a polynomial

Is continuous at 3 b/c polynomials are continuous for all x

4) $g(t) = \sqrt{t-4}$

then $g(3) = \sqrt{3-4} = \sqrt{-1}$ is an even root of a negative number

$\Rightarrow$ $g(t)$ is not continuous (or defined) at $t=3$

10) $f(x) = \dfrac{21-7x}{x-3} = \dfrac{-7(x-3)}{(x-3)}$ $\Rightarrow$ 3 is a root of the top and bottom of this rational function $\Rightarrow$ there is a hole at 3

$f(x)$ is not continuous at 3 because

$\lim\limits_{x\to 3} f(x) = -7$ but $f(3)$ is not defined

$\Rightarrow$ $\lim\limits_{x\to 3} f(x) \neq f(3)$

20) $g(\theta) = \dfrac{\sin\theta}{\theta}$ we know that at $\theta = 0$ we have a $\frac{0}{0}$ situation

$\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$ (special Limit theorem for trig functions)

$\Rightarrow$ we need $\boxed{g(0) = 1}$ so that $\lim\limits_{\theta\to 0} g(\theta) = g(0) = 1$

21) $H(t) = \dfrac{\sqrt{t} - 1}{t - 1}$ Note: at $t = 1$ we have a $\frac{0}{0}$ situation

Note: $t - 1 = (\sqrt{t} + 1)(\sqrt{t} - 1)$

$$\lim_{t \to 1} \frac{\sqrt{t} - 1}{t - 1} = \lim_{t \to 1} \frac{(\sqrt{t} - 1)}{(\sqrt{t} + 1)(\sqrt{t} - 1)} = \lim_{t \to 1} \frac{1}{\sqrt{t} + 1} = \frac{1}{1 + 1} = \frac{1}{2}$$

$\Rightarrow$ we need $H(1) = \frac{1}{2}$ so that

$$\lim_{t \to 1} H(t) = H(1) = \frac{1}{2}$$

24) $f(x) = \dfrac{3x + 7}{(x - 30)(x - \pi)}$ is a rational function, we divide by zero at $x = 30$, $x = \pi$, neither of which are roots of $3x + 7$

$\Rightarrow$ we have vertical asymptotes at $x = 30$, $x = \pi$

$\Rightarrow$ $f(x)$ is discontinuous at $x = 30$, $x = \pi$

32) $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } x > 1 \end{cases}$

x, x^2, $2 - x$ are all polynomials and are continuous everywhere so any discontinuities will come where we change from one function to another $(x = 0,\ x = 1)$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x = 0, \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0$$

$$\Rightarrow \lim_{x \to 0} f(x) = 0$$

$f(0) = (0)^2 = 0$ $\Rightarrow$ $f(x)$ is continuous at $x = 0$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = (1)^2 = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x) = 2-1 = 1$$

$$\Rightarrow \lim_{x\to 1} f(x) = 1$$

$f(1) = (1)^2 = 1 \quad \Rightarrow \quad f(x)$ is continuous at $x=1$

$\Rightarrow f(x)$ is continuous everywhere, there are no discontinuities.

<u>Section 2.1:</u>

10) $y = x^3 - 3x$

$$y'(x) = \lim_{h\to 0} \frac{(x+h)^3 - 3(x+h) - (x^3 - 3x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{3x} - 3h - \cancel{x^3} + \cancel{3x}}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 - 3$$

$$y'(x) = 3x^2 - 3$$

The slope of the tangent line to the curve = the derivative evaluated at the point

$\Rightarrow y'(-2) = 3(-2)^2 - 3 = \boxed{9}$

$y'(-1) = 3(-1)^2 - 3 = \boxed{0}$

$y'(0) = 3(0)^2 - 3 = \boxed{-3}$

$y'(1) = 3(1)^2 - 3 = \boxed{0}$

$y'(2) = 3(2)^2 - 3 = \boxed{9}$

13) Body Falls $16t^2$ feet in t seconds

$\Rightarrow \quad d(t) = 16t^2 \qquad d(t)$ = distance fallen as a function of time

a) $d(0) = 0, \quad d(1) = 16(1)^2 = 16$

$\Rightarrow \quad d(1) - d(0) = \boxed{16}$ feet fallen between $t=0$, and $t=1$

b) $d(1) = 16, \quad d(2) = 16(2)^2 = 64$

$\Rightarrow \quad d(2) - d(1) = 64 - 16 = \boxed{48}$ feet fallen between $t=1, t=2$

c) $\text{Average velocity} = \dfrac{\text{distance travelled}}{\text{time taken}} = \dfrac{d(3) - d(2)}{3-2} = \dfrac{144-64}{3-2}$

$d(3) = 16(3)^2 = 144 \qquad\qquad = \boxed{80}$ feet per second

$d(2) = 16(2)^2 = 64$

15) Suppose that an object moves along a coordinate line (the x-axis) so that its directed distance from the origin after t seconds is $\sqrt{2t+1}$ feet

$x(t) = \sqrt{2t+1}$

a) Instantaneous velocity $= \dfrac{d}{dt}[x(t)]$ = time derivative of the distance function

$$\frac{d}{dt}\left[\sqrt{2t+1}\right] = \lim_{h\to 0} \frac{\left(\sqrt{2(t+h)+1}\right) - \sqrt{2t+1}}{h}$$

note $a^2 - b^2 = (a+b)(a-b)$

$\Rightarrow a - b = (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$

$$= \lim_{h\to 0} \frac{\left[\sqrt{2t+2h+1} - \sqrt{2t+1}\right]}{h} \frac{\left[\sqrt{2t+2h+1} + \sqrt{2t+1}\right]}{\left[\sqrt{2t+2h+1} + \sqrt{2t+1}\right]}$$

$$= \lim_{h\to 0} \frac{(2t+2h+1)-(2t+1)}{h\left(\sqrt{2t+2h+1}+\sqrt{2t+1}\right)} = \lim_{h\to 0} \frac{\cancel{2t}+2h+\cancel{1}-\cancel{2t}-\cancel{1}}{h\left(\sqrt{2t+2h+1}+\sqrt{2t+1}\right)}$$

$$= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}\left(\sqrt{2t+2h+1}+\sqrt{2t+1}\right)} = \lim_{h\to 0} \frac{2}{\sqrt{2t+2h+1}+\sqrt{2t+1}}$$

$$= \frac{2}{\sqrt{2t+2(0)+1}+\sqrt{2t+1}} = \frac{2}{\sqrt{2t+1}+\sqrt{2t+1}} = \frac{2}{2\sqrt{2t+1}}$$

$$x'(t) = \frac{1}{\sqrt{2t+1}} = v(t)$$

if $t = \alpha$, then $v(t) = \boxed{v(\alpha) = \frac{1}{\sqrt{2\alpha+1}}, \ \alpha > 0}$

b) when will $v(t) = \frac{1}{2} \frac{ft}{sec}$?

$$v(t) = \frac{1}{2} \Rightarrow \frac{1}{\sqrt{2t+1}} = \frac{1}{2}$$

$$\Rightarrow \sqrt{2t+1} = 2$$

$$2t+1 = 4$$

$$2t = 3$$

$$\boxed{t = \frac{3}{2}}$$

Section 2.2: 1, 4, 9, 14, 22, 25, 27, 39, 42

1) $f'(1)$ if $f(x) = x^2$

$$\Rightarrow f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2 - (1)^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{1^2} + 2h + h^2 - \cancel{1^2}}{h} = \lim_{h\to 0} \frac{2h + h^2}{h} = \lim_{h\to 0} 2 + h = \boxed{2}$$

4) $f'(4)$ if $f(s) = \frac{1}{s-1}$

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\frac{1}{4+h-1} - \frac{1}{4-1}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h} = \lim_{h\to 0} \frac{\frac{1}{3+h}\cdot\frac{3}{3} - \frac{1}{3}\cdot\frac{3+h}{3+h}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3}{3(3+h)} - \frac{3+h}{3(3+h)}}{h} = \lim_{h\to 0} \frac{\frac{3-3-h}{3(3+h)}}{\frac{h}{1}}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{3(3+h)\cancel{h}} = \lim_{h\to 0} \frac{-1}{3(3+h)} = \frac{-1}{3(3+0)} = \boxed{\frac{-1}{9}}$$

9) $f(x) = ax^2 + bx + c$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a(x+h)^2 + b(x+h) + c - (ax^2 + bx + c)}{h}$$

$$= \lim_{h\to 0} \frac{a(\cancel{x^2} + 2xh + h^2) + \cancel{bx} + bh + \cancel{c} - \cancel{ax^2} - \cancel{bx} - \cancel{c}}{h}$$

$$= \lim_{h\to 0} \frac{2axh + ah^2 + bh}{h} = \lim_{h\to 0} 2ax + ah + b$$

$$= \boxed{2ax + b}$$

14) $S(x) = \dfrac{1}{x+1}$

$$S'(x) = \lim_{h\to 0} \frac{S(x+h) - S(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{x+h+1}\cdot\frac{x+1}{x+1} - \frac{1}{x+1}\cdot\frac{x+h+1}{x+h+1}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x+1}{(x+1)(x+h+1)} - \frac{x+h+1}{(x+1)(x+h+1)}}{h} = \lim_{h\to 0} \frac{\frac{x+1-(x+h+1)}{(x+1)(x+h+1)}}{\frac{h}{1}}$$

$$= \lim_{h\to 0} \frac{\cancel{x} + \cancel{1} - \cancel{x} - h - \cancel{1}}{h(x+1)(x+h+1)} = \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(x+1)(x+h+1)}$$

$$= \lim_{h\to 0} \frac{-1}{(x+1)(x+h+1)} = \frac{-1}{(x+1)(x+0+1)} = \boxed{\frac{-1}{(x+1)^2}}$$

22) $H(x) = \sqrt{x^2+4}$

$$H'(x) = \lim_{h\to 0} \frac{H(x+h) - H(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)^2+4} - \sqrt{x^2+4}}{h}$$

$$= \lim_{h\to 0} \frac{\left[\sqrt{(x+h)^2+4} - \sqrt{x^2+4}\right]}{h} \frac{\left[\sqrt{(x+h)^2+4} + \sqrt{x^2+4}\right]}{\left[\sqrt{(x+h)^2+4} + \sqrt{x^2+4}\right]}$$

$$= \lim_{h\to 0} \frac{(x+h)^2+4 - (x^2+4)}{h\left[\sqrt{(x+h)^2+4} + \sqrt{x^2+4}\right]} = \lim_{h\to 0} \frac{\cancel{x^2} + 2hx + h^2 + \cancel{4} - \cancel{x^2} - \cancel{4}}{h\left[\sqrt{(x+h)^2+4} + \sqrt{x^2+4}\right]}$$

$$= \lim_{h\to 0} \frac{2\cancel{h}x + h^{\cancel{2}}}{\cancel{h}\left[\sqrt{(x+h)^2+4} + \sqrt{x^2+4}\right]} = \lim_{h\to 0} \frac{2x+h}{\sqrt{(x+h)^2+4} + \sqrt{x^2+4}}$$

$$= \frac{2x+0}{\sqrt{(x+0)^2+4} + \sqrt{x^2+4}} = \frac{2x}{\sqrt{x^2+4}+\sqrt{x^2+4}} = \frac{2x}{2\sqrt{x^2+4}}$$

$$= \boxed{\frac{x}{\sqrt{x^2+4}}}$$

25) $f(x) = \dfrac{x}{x-5}$

$$f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x} = \lim_{t\to x} \frac{\frac{t}{t-5} - \frac{x}{x-5}}{t-x}$$

$$= \lim_{t\to x} \frac{\frac{t}{t-5}\cdot\frac{x-5}{x-5} - \frac{x}{x-5}\cdot\frac{t-5}{t-5}}{t-x}$$

$$= \lim_{t\to x} \frac{\frac{t(x-5)}{(t-5)(x-5)} - \frac{x(t-5)}{(t-5)(x-5)}}{t-x} = \lim_{t\to x} \frac{\frac{tx-5t-(xt-5x)}{(t-5)(x-5)}}{\frac{(t-x)}{1}}$$

$$= \lim_{t\to x} \frac{\cancel{tx} - 5t - \cancel{tx} + 5x}{(t-x)(t-5)(x-5)} = \lim_{t\to x} \frac{-5\cancel{(t-x)}}{\cancel{(t-x)}(t-5)(x-5)}$$

$$= \lim_{t\to x} \frac{-5}{(t-5)(x-5)} = \frac{-5}{(x-5)(x-5)} = \boxed{\frac{-5}{(x-5)^2}}$$

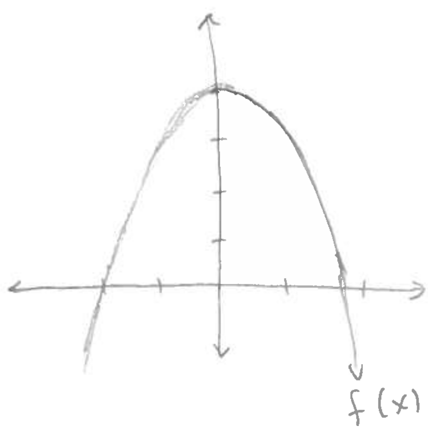
27) $\lim_{h \to 0} \dfrac{2(5+h)^3 - 2(5)^3}{h} = \lim_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$

$\Rightarrow \quad 2(5+h)^3 = f(c+h) \quad , \quad 2(5)^3 = f(c)$

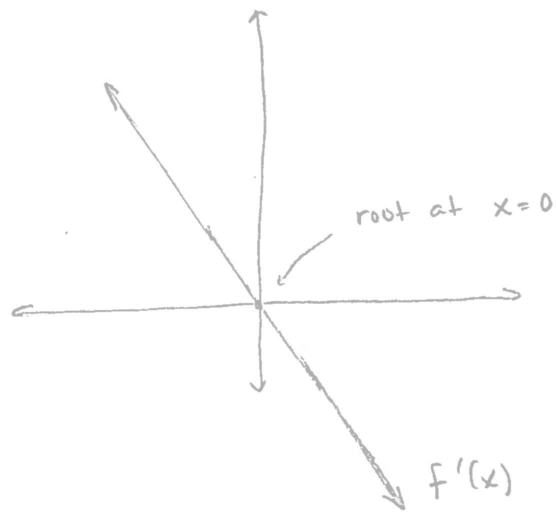
$\Rightarrow \quad c = 5 \; , \quad f(c) = 2c^3$

$\Rightarrow \quad f(x) = 2x^3 \quad$ at $\quad x = 5$

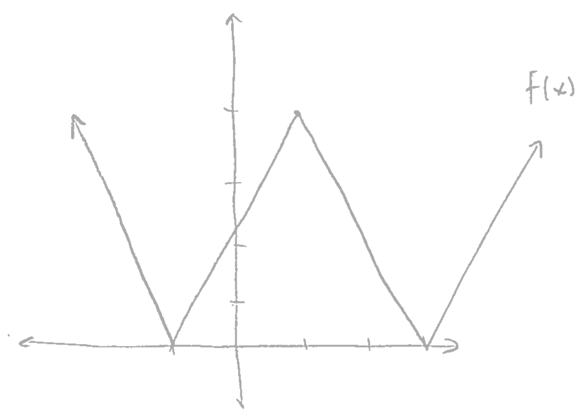
39)



$\Rightarrow$



42)



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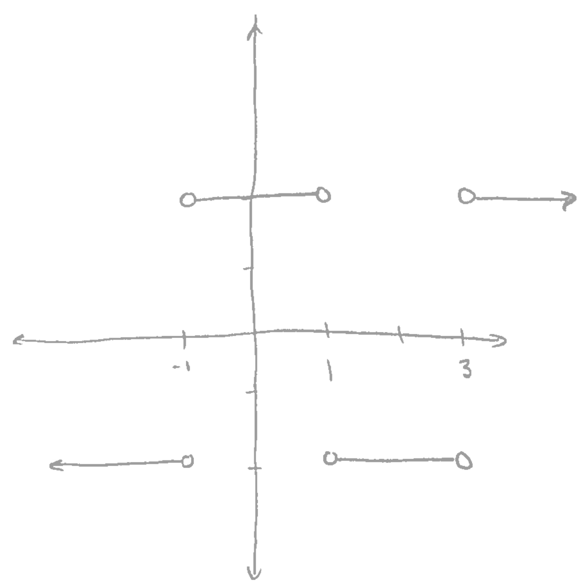

slope $= \pm 2$,

jumps at the corners