# 1170:Lab9 

November 5th, 2013

## Goals For This Week

In this ninth lab we will explore our second numerical method, Euler's method for solving Differential Equations

- Motivation
- Describing Euler's Method
- Code for Euler's Method


## 1 Motivation

Differential equations are a vital subject in applied mathematics involving the relationship between a function and its derivative. Consider for example, that the velocity of an object $\mathrm{v}(\mathrm{t})$ is some known function. Velocity is the derivative of position. As such it must be that

$$
\begin{equation*}
\frac{d p}{d t}=v(t) \tag{1}
\end{equation*}
$$

We now have a differential equation relating the position of an object to its velocity as a function of time. If the velocity is some known function than we can solve this equation for $p(t)$ which gives us the position of the object for all time. Depending on the function $v(t)$ this equation may be difficult if not impossible to solve analytically. As such we would like to consider approximations to the solution of the differential equation.

## 2 Euler's Method for solving Differential Equations

Last lab we introduced our first numerical scheme, Newton's Method. With Newton's method our goal was to try to find the root of a function by taking tangent lines of the function near to its root as an approximation and then finding the root of the tangent line.

For Euler's method we would like to solve the differential equation for $\mathrm{p}(\mathrm{t})$, but the only information that we have about $\mathrm{p}(\mathrm{t})$ is that we know what its derivative $\mathrm{v}(\mathrm{t})$ is, and possibly some initial condition (where the function intersects a point). As our tangent line approximation needs knowledge of the derivative and a point of the function our goal is then to approximate the function locally as its tangent line.

For example suppose $v(t)=1.5$, and that the object starts at $p(0)=0$. Then

$$
\begin{align*}
& \frac{d p}{d t}=1.5  \tag{2}\\
& p(0)=0 \tag{3}
\end{align*}
$$

We know that at $t=0$ the location of the object is $p=0$. Its derivative is $v(0)=1.5$. Therefore the tangent line approximation

$$
\begin{equation*}
p(0+t) \approx p(0)+p^{\prime}(0)(t-0) p(0+t) \approx 0+1.5(t) p(0+t) \approx 1.5 t \tag{4}
\end{equation*}
$$

From earlier however we know that a tangent line approximation is only a good approximation of a function very close to the point. As such we are going to define $t=\delta t$ where $\delta t$ is some small number. Then

$$
\begin{align*}
p(0+\delta t) & \approx 1.5 \delta t  \tag{5}\\
p(\delta t) & \approx 1.5 \delta t \tag{6}
\end{align*}
$$

We now have an approximation for $\mathrm{p}(\mathrm{t})$ up from time $\mathrm{t}=0$ up to time $t=\delta t$. We now want to do the same thing again, but this time using the tangent line at $(\delta t, p(\delta t))$

$$
\begin{array}{r}
p(\delta t+\delta t) \approx p(\delta t)+p^{\prime}(\delta t)(2 \delta t-\delta t) \\
p(2 \delta t) \approx(1.5 \delta t)+v(\delta t)(\delta t) \\
p(2 \delta t) \approx 1.5 \delta t+1.5(\delta t) \\
p(2 \delta t) \approx 3 \delta t \tag{10}
\end{array}
$$

We will than continue in this manner until we reach the time that we want to stop at.
To ground this example let us let $\delta t=1$
Then:

$$
\begin{array}{r}
p(1)=1.5(1)=1.5 \\
p(2)=3(1)=3 \tag{12}
\end{array}
$$

## 3 R Code for Euler's Method

First we are going to define some variables for the equation:

$$
\begin{array}{r}
\frac{d f}{d t}=g(t) \\
f(a)=f a(\text { initialvalue }) \tag{14}
\end{array}
$$

We will call:
$\mathrm{a}=$ the left endpoint, or our initial value
$\mathrm{b}=$ the right endpoint, or out terminal value
$\mathrm{n}=$ the number of time steps we want to make
delt $=$ timestep $\delta t$
$t=a$ sequence of the time values
$\mathrm{f}=\mathrm{a}$ sequence of the function values
$\mathrm{g}=$ the derivative function
Consider the following differential equation which we want to solve over the interval $\mathrm{t}=(0,10)$

$$
\begin{gather*}
\frac{d f}{d t}=e^{t}  \tag{15}\\
f(0)=0 \tag{16}
\end{gather*}
$$

Then:
$\mathrm{a}=0$
$b=10$
$\mathrm{n}=10$
delt $=(\mathrm{b}-\mathrm{a}) / \mathrm{n}$
$\mathrm{t}=\operatorname{seq}(1, \mathrm{n})$ (preallocate the time values)

```
\(\mathrm{f}=\operatorname{seq}(1, \mathrm{n})\) (preallocate the f values)
```

$\mathrm{g}=e^{t}$
We will now iterate the code in a for loop:

```
g<-function(t){exp(t)}
a<-0
b<-10
n<-10
t<-seq(1,n)
f<-seq(1,n)
t[1]<-0
f[1]<-0
delt<-(b-a)/n
for (i in 2:(n+1)) {t[i]<-t[i-1]+delt
f[i]<-g(t[i-1])*delt+f[i-1]}
plot(t,f,type='l')
```

Now lets see how it compares to the solution $f(x)=e^{x}-1$.

```
x<-seq(0,10,.01)
sol<-function(x){exp(x)-1}
lines(x,sol(x))
```

Change n to be 100 and rerun the code. Does this give us a better approximation of the solution?

## 4 Assignment for this week

Consider the differential equation $\frac{d f}{d t}=e^{\sin (t)}$, with initial condition $\mathrm{f}(0)=0$.

1. Write your own Euler's Method code to approximate the solution to this differential equation.
2. Plot the resulting solutions for $n=10, n=100$, and $n=1000$
3. What would cause Euler's method to perform poorly. What types of derivative functions would cause there to be large error. Suggest an improvement that could be made to Euler's method that would help to alleviate this problem.
