## 1170:Lab12

November 26th, 2013

## Goals For This Week

In this twelfth lab we will explore autonomous differential equations, and how to use Euler's method to approximate their solutions.

## 1 Autonomous Differential Equations

Suppose we have a differential equation:

$$
\begin{align*}
& \frac{d f}{d t}=g(f)  \tag{1}\\
& f(a)=f_{a} \tag{2}
\end{align*}
$$

Notice that the right hand side of the equation depends on the function $f$, not on the variable $t$. We say that such an equation is a (time) autonomous differential equation.

Consider for example:

$$
\begin{equation*}
\frac{d H}{d t}=\alpha(A-H) \tag{3}
\end{equation*}
$$

This equation is called Newton's law of cooling. First note that the right hand side of the differential equation depends on the state variable H , and not on the time. Therefore Newton's law of cooling is a time autonomous differential equation. Here $\alpha$ is a physical parameter with units $1 /$ time, and A is called the ambient temperature(or room temperature).

## 2 Exploring Newton's Law of Cooling with a phase plane

Recall:

$$
\begin{equation*}
\frac{d H}{d t}=\alpha(A-H) \tag{4}
\end{equation*}
$$

since $\alpha$ is a positive constant, what happens to the temperature H , if $H>A$ ? What about if $H<A$ ?
By qualitatively exploring the equation with a phase portrait we can get a feel for what the solution curves will looks like.

What will happen to the temperature H as time approaches infinity?

## 3 Exploring Newton's Law of Cooling with Euler's Method

Suppose that $\mathrm{A}=50$, and the $\alpha=0.1$ and that $\mathrm{H}(0)=30$ Then we can modify Euler's method for a autonomous differential equation in the following way.

First let us pick a time step size, $\delta t=0.25$, and suppose we want to estimate the temperature after 1 second, $\mathrm{H}(1)$.

1. First we start at the initial condition point, $\mathrm{H}(0)=30$. We then find the tangent line with this base point $\mathrm{t}=0$.

$$
\begin{array}{r}
H(0+\delta t)=H(0)+H^{\prime}(0) * \delta t \\
H(\delta t)=H(0)+(\alpha *(A-H(0))) * \delta t \\
H(0.25)=30+(0.1) *(50-30) *(0.25) \\
H(0.25)=30+.5=30.5
\end{array}
$$

2. Now we use $\mathrm{H}(0.25)=30.5$ as an initial point to find $\mathrm{H}(0.5)$

$$
\begin{array}{r}
H(0.25+\delta t)=H(0.25)+H^{\prime}(0.25) * \delta t \\
H(0.5)=30.5+(0.1) *(50-30.5) *(0.25) \\
H(0.5)=30.9875
\end{array}
$$

3. To find $\mathrm{H}(0.75)$

$$
\begin{array}{r}
H(0.5+\delta t)=H(0.5)+H^{\prime}(0.5) * \delta t \\
H(0.75)=30.9875+(0.1) *(50-30.9875) *(0.25) \\
H(0.75)=31.463
\end{array}
$$

4. Finally to find $H(1)$

$$
\begin{array}{r}
H(0.75+\delta t)=H(0.75)+H^{\prime}(0.75) * \delta t \\
H(1)=31.463+(0.1) *(50-31.463) *(0.25) \\
H(1)=31.926
\end{array}
$$

What can we say about the temperature as a function of time?
Notice that Euler's Method worked the same was as before, but H' was a function of H, not of t . Accordingly we can change our code for Euler's method by changing:

```
f[i]<-g(t[i-1])*delt+f[i-1]
to
f[i]<-g(H[i-1])*delt+f[i-1]
```

Giving us the following Euler's Method code:

```
g<-function(H) {0.1*(50-H)}
a<-0
b<-50
n<-1000
t<-seq(1,n+1)
f<-seq(1,n+1)
t[1]<-a
f[1]<-30
delt<-(b-a)/n
for (i in 2:(n+1)) {t[i]<-t[i-1]+delt
f[i]<-g(f[i-1])*delt+f[i-1]}
plot(t,f,type='l')
```


## 4 Assignment for this week

Consider the following differential equation which models the number of individuals infected with a disease, I:

$$
\begin{array}{r}
\frac{d I}{d t}=\alpha I(1-I)-\mu I \\
I(0)=0.1 \tag{6}
\end{array}
$$

1. First assume that $\alpha>\mu$. This means that the infection rate $\alpha$ is greater than the recovery rate $\mu$. Solve for any equilibria (hint: when $\frac{d I}{d t}=0$ ) by hand. Then rewrite Euler's method for this differential equation and solve from $\mathrm{t}=0$ to $\mathrm{t}=15$ using $\alpha=2, \mu=1$. Finally add in the direction arrows to the phase portrait, you can add them to the plot from Euler's method if you want.
2. Now assume that $\mu>\alpha$ and repeat the steps from problem 1 with $\mu=2, \alpha=1$.
