# 1170:Lab12

#### November 26th, 2013

#### Goals For This Week

In this twelfth lab we will explore autonomous differential equations, and how to use Euler's method to approximate their solutions.

## **1** Autonomous Differential Equations

Suppose we have a differential equation:

$$\frac{df}{dt} = g(f) \tag{1}$$

$$(a) = f_a \tag{2}$$

Notice that the right hand side of the equation depends on the function f, not on the variable t. We say that such an equation is a (time) autonomous differential equation.

Consider for example:

$$\frac{dH}{dt} = \alpha (A - H) \tag{3}$$

This equation is called Newton's law of cooling. First note that the right hand side of the differential equation depends on the state variable H, and not on the time. Therefore Newton's law of cooling is a time autonomous differential equation. Here  $\alpha$  is a physical parameter with units 1/time, and A is called the ambient temperature(or room temperature).

# 2 Exploring Newton's Law of Cooling with a phase plane

Recall:

$$\frac{dH}{dt} = \alpha(A - H) \tag{4}$$

since  $\alpha$  is a positive constant, what happens to the temperature H, if H > A? What about if H < A?

By qualitatively exploring the equation with a phase portrait we can get a feel for what the solution curves will looks like.

What will happen to the temperature H as time approaches infinity?

### 3 Exploring Newton's Law of Cooling with Euler's Method

Suppose that A=50, and the  $\alpha = 0.1$  and that H(0)=30 Then we can modify Euler's method for a autonomous differential equation in the following way.

First let us pick a time step size,  $\delta t = 0.25$ , and suppose we want to estimate the temperature after 1 second, H(1).

1. First we start at the initial condition point, H(0)=30. We then find the tangent line with this base point t=0.

$$H(0 + \delta t) = H(0) + H'(0) * \delta t$$
$$H(\delta t) = H(0) + (\alpha * (A - H(0))) * \delta t$$
$$H(0.25) = 30 + (0.1) * (50 - 30) * (0.25)$$
$$H(0.25) = 30 + .5 = 30.5$$

2. Now we use H(0.25)=30.5 as an initial point to find H(0.5)

 $H(0.25 + \delta t) = H(0.25) + H'(0.25) * \delta t$  H(0.5) = 30.5 + (0.1) \* (50 - 30.5) \* (0.25)H(0.5) = 30.9875

3. To find H(0.75)

$$H(0.5 + \delta t) = H(0.5) + H'(0.5) * \delta t$$
  
$$H(0.75) = 30.9875 + (0.1) * (50 - 30.9875) * (0.25)$$
  
$$H(0.75) = 31.463$$

4. Finally to find H(1)

$$H(0.75 + \delta t) = H(0.75) + H'(0.75) * \delta t$$
  

$$H(1) = 31.463 + (0.1) * (50 - 31.463) * (0.25)$$
  

$$H(1) = 31.926$$

What can we say about the temperature as a function of time?

Notice that Euler's Method worked the same was as before, but H' was a function of H, not of t. Accordingly we can change our code for Euler's method by changing:

f[i]<-g(t[i-1])\*delt+f[i-1]
to
f[i]<-g(H[i-1])\*delt+f[i-1]</pre>

Giving us the following Euler's Method code:

```
g<-function(H){0.1*(50-H)}
a<-0
b<-50
n<-1000
t<-seq(1,n+1)
f<-seq(1,n+1)
t[1]<-a
f[1]<-30
delt<-(b-a)/n
for (i in 2:(n+1)) {t[i]<-t[i-1]+delt
f[i]<-g(f[i-1])*delt+f[i-1]}
plot(t,f,type='1')</pre>
```

## 4 Assignment for this week

Consider the following differential equation which models the number of individuals infected with a disease, I:

$$\frac{dI}{dt} = \alpha I(1-I) - \mu I \tag{5}$$

$$I(0) = 0.1$$
 (6)

- 1. First assume that  $\alpha > \mu$ . This means that the infection rate  $\alpha$  is greater than the recovery rate  $\mu$ . Solve for any equilibria (hint: when  $\frac{dI}{dt} = 0$ ) by hand. Then rewrite Euler's method for this differential equation and solve from t=0 to t=15 using  $\alpha = 2$ ,  $\mu = 1$ . Finally add in the direction arrows to the phase portrait, you can add them to the plot from Euler's method if you want.
- 2. Now assume that  $\mu > \alpha$  and repeat the steps from problem 1 with  $\mu = 2, \alpha = 1$ .