Counting II

Sometimes we will want to choose k objects from a set of n objects, and we won't be interested in ordering them. For example, if you are leaving for vacation and you want to pack your suitcase with three of the seven pairs of shorts that you own, then it doesn't matter in which order you pack the shorts. All that matters is which three pairs you pack.

n choose k

The number of different ways that k objects can be chosen from a set of n objects (when order doesn't matter) is called *n choose k*. It is written in symbol form as $\binom{n}{k}$.

Examples.

• There are four different ways that one letter can be chosen from the set of four letters $\{\mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{a}\}$. One way is to choose the letter $\mathbf{e}$. Alternatively, you could also choose the letter $\mathbf{f}$, or the letter $\mathbf{g}$, or the letter $\mathbf{a}$.

Since there are 4 options for choosing one object from a set of 4 objects, we have $\binom{4}{1} = 4$.

• Below is a list of all the possible ways that 2 numbers can be chosen from the set of four numbers $\{3, 7, 2, 9\}$. There are six different ways. Thus, $\binom{4}{2} = 6$.

$3, 7$	$3, 2$	$3, 9$
$7, 2$	$7, 9$	$2, 9$

General formula

To say that we are choosing and ordering k objects from a set of n objects is to say that we are performing 2 separate tasks. First is the task of choosing k objects from the set of n objects, and the number of ways to perform that task is $\binom{n}{k}$. Second is the task of ordering the k objects after we've chosen them. There are $k!$ ways to order k objects.

Let's repeat that. To choose and order k objects: First, choose the k objects, then order the k objects you chose. Options multiply, so the total number of ways that we can choose and order k objects from a set of n objects is $\binom{n}{k}k!$.

We saw in the previous chapter that there are exactly $\frac{n!}{(n-k)!}$ ways to choose and order k objects from a set of n objects. Therefore,

$$\binom{n}{k} k! = \frac{n!}{(n-k)!}$$

Dividing the previous equation by $k!$:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Examples.

- There are $\binom{7}{3}$ different ways to choose which 3 of the 7 pairs of shorts that you will take on your vacation.

$$\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7(6)(5)(4!)}{3!4!} = \frac{7(6)(5)}{6} = 7(5) = 35$$

- How many 5 card poker hands are there if you play with a standard deck of 52 cards?

You're counting the number of different collections of 5 cards that can be taken from a set of 52 cards. This number is

$$\binom{52}{5} = \frac{52!}{5!(52-5)!} = \frac{52(51)(50)(49)(48)(47!)}{5!\,47!} = \frac{52(51)(50)(49)(48)}{5!}$$
$$= \frac{52(51)(50)(49)(48)}{120} = 2,598,960$$

- You are at a DVD rental store. You want to rent 3 DVDs. The store has 3287 different DVDs to choose from. There are $\binom{3287}{3}$ different collections of three movies that you could rent.

$$\binom{3287}{3} = \frac{3287!}{3!(3287-3)!} = \frac{3287(3286)(3285)(3284!)}{3!\,3284!} = \frac{3287(3286)(3285)}{3!}$$
$$= \frac{35,481,554,370}{6} = 5,913,592,395$$

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Pascal's Triangle

We can arrange the numbers $\binom{n}{k}$ into a triangle.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & \binom{0}{0} & & & & & & & \\
 & & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & & \\
 & & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & & \\
 & & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & & \\
 & & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & & \\
 & & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & & \\
 & \binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6} & \\
\binom{7}{0} & & \binom{7}{1} & & \binom{7}{2} & & \binom{7}{3} & & \binom{7}{4} & & \binom{7}{5} & & \binom{7}{6} & & \binom{7}{7}
\end{array}
$$

In each row, the "top" number of $\binom{n}{k}$ is the same. The "bottom" number of $\binom{n}{k}$ is the same in each upward slanting diagonal. The triangle continues on forever. The first 8 rows are shown above.

This is called Pascal's triangle. It is named after a French mathematician who discovered it. It had been discovered outside of Europe centuries earlier by Chinese mathematicians. Modern mathematics began in Europe, so it's traditions and stories tend to promote the exploits of Europeans over others.

Some values of $\binom{n}{k}$ to start with

$\binom{n}{n}$ is the number of different ways you can select n objects from a set of n objects. There is only one way to take everything – you just take everything – so $\binom{n}{n} = 1$.

Similarly, there is only one way to take nothing from a set – just take nothing, that's your only option. The number of ways you can select nothing, a.k.a. 0 objects, from a set is $\binom{n}{0}$. That means $\binom{n}{0} = 1$.

Now we can fill in the values for $\binom{n}{n}$ and $\binom{n}{0}$ into Pascal's triangle.

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad \binom{2}{1} \quad 1 \\
1 \quad \binom{3}{1} \quad \binom{3}{2} \quad 1 \\
1 \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad 1 \\
1 \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad 1 \\
1 \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad 1 \\
1 \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad 1
\end{array}
$$

There are n different ways to choose 1 object from a set that has n objects. Thus, $\binom{n}{1} = n$.

Similarly, there are exactly n different options for choosing $n-1$ objects from a set of n objects. To see this, notice that deciding which of the $n-1$ objects that you will take from a set of n objects is the same as deciding which 1 object you will leave behind. So the number of ways you can take $n-1$ objects is the same as the number of ways you can leave 1 object. That is to say, $\binom{n}{n-1} = \binom{n}{1}$. Hence, $\binom{n}{n-1} = n$

Once we fill in this new information on Pascal's triangle it looks like

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad \binom{4}{2} \quad 4 \quad 1 \\
1 \quad 5 \quad \binom{5}{2} \quad \binom{5}{3} \quad 5 \quad 1 \\
1 \quad 6 \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad 6 \quad 1 \\
1 \quad 7 \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad 7 \quad 1
\end{array}
$$

Before moving on, let's look back at the last rule that we found: $\binom{n}{n-1} = \binom{n}{1}$. The argument we gave there generalizes, in that taking k objects from a set of n objects is the same as leaving $n-k$. Therefore,

$$\binom{n}{k} = \binom{n}{n-k}$$

What we saw earlier in the form $\binom{n}{n-1} = \binom{n}{1}$ was the special case of the formula $\binom{n}{k} = \binom{n}{n-k}$ when $k = n-1$.

Add the two numbers above to get the number below

Suppose that you have a set of n different rocks: 1 big red brick, and $n-1$ different little blue marbles. How many different ways are there to choose $k+1$ rocks from the set of n rocks?

Any collection of $k+1$ rocks either includes the big red brick, or it doesn't.

Let's first look at those collections of $k+1$ objects that *do* contain a big red brick. One of the $k+1$ objects we will choose is a big red brick. That's a given. That means that all we have to do is decide which k of the little blue marbles we want to choose along with the big red brick to make up our collection of $k+1$ objects. There are $\binom{n-1}{k}$ different ways we could choose k marbles from the total number of $n-1$ little blue marbles. Thus, there are $\binom{n-1}{k}$ different ways we could choose a set of $k+1$ objects from our set of n rocks if we know that one of the objects we will choose is a big red brick.

Now let's look at those collections of $k+1$ objects that *don't* contain the big red brick. Then all $k+1$ objects that we will choose are little blue marbles. There are $n-1$ little blue marbles, and the number of different ways we could choose $k+1$ of the $n-1$ little blue marbles is $\binom{n-1}{k+1}$.

Any collection of $k+1$ rocks either includes the big red brick, or it doesn't. So to find the number of ways that we could choose $k+1$ objects, we just have to add the number of possibilities that contain a big red brick, to the number of possibilities that don't contain a big red brick. That formula is

$$\binom{n}{k+1} = \binom{n-1}{k} + \binom{n-1}{k+1}$$

If $n = 6$ and $k = 2$, then the above formula says that $\binom{6}{3} = \binom{5}{2} + \binom{5}{3}$. Looking at Pascal's triangle, you'll see that $\binom{5}{2}$ and $\binom{5}{3}$ are the two numbers that are just above the number $\binom{6}{3}$.

Change the values of n and k and check that the above formula always indicates that to find a number in Pascal's triangle, just sum the two numbers that are directly above it.

For example,

$$\binom{2}{1} = \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2$$

and

$$\binom{5}{4} = \binom{4}{3} + \binom{4}{4} = 4 + 1 = 5$$

We know the natural numbers that are at the very tip of Pascal's triangle. To find the rest of the numbers in Pascal's triangle, we can let that knowledge trickle down the triangle using this latest formula that any number in the triangle is the sum of the two numbers above it.

$$\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&&\\
&&&&&&1&&1&&&&&&\\
&&&&&1&&2&&1&&&&&\\
&&&&1&&3&&3&&1&&&&\\
&&&1&&4&&6&&4&&1&&&\\
&&1&&5&&10&&10&&5&&1&&\\
&1&&6&&15&&20&&15&&6&&1&\\
1&&7&&21&&35&&35&&21&&7&&1
\end{array}$$

Notice that Pascal's triangle is the same if you read it left-to-right, or right-to-left. Convince yourself that this is a consequence of the formula $\binom{n}{k} = \binom{n}{n-k}$.

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Binomial Theorem

For $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

Example. We can use the binomial theorem and Pascal's triangle to write out the product $(x+y)^3$. The binomial theorem states that

$$\begin{aligned}(x+y)^3 &= \sum_{i=0}^{3} \binom{3}{i} x^{3-i} y^i \\ &= \binom{3}{0} x^{3-0} y^0 + \binom{3}{1} x^{3-1} y^1 + \binom{3}{2} x^{3-2} y^2 + \binom{3}{3} x^{3-3} y^3 \\ &= \binom{3}{0} x^3 + \binom{3}{1} x^2 y + \binom{3}{2} x y^2 + \binom{3}{3} y^3\end{aligned}$$

The numbers $\binom{3}{0}$, $\binom{3}{1}$, $\binom{3}{2}$, and $\binom{3}{3}$ make up the fourth row of Pascal's triangle, and we can see from the triangle that they equal 1, 3, 3, and 1 respectively. Therefore,

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Binomial coefficients. Because of the binomial theorem, numbers of the form $\binom{n}{k}$ are called *binomial coefficients.*

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Exercises

1.) A small country has just undergone a revolution and they commission you to design a new flag for them. The want exactly 3 colors used in their flag, and they have given you 7 colors of cloth that you are allowed to use. How many different color combinations do you have to decide between?

2.) A sports team is selling season ticket plans. They have 15 home games in a season, and they allow people to purchase tickets for any combination of 7 home games. How many ways are there to choose a collection of 7 home games?

3.) A bagel shop asks customers to create a “Baker’s Dozen Variety Pack” by choosing 13 different types of bagels. If the shop has 20 different kinds of bagels to choose between, then how many different variety packs does a customer have to choose from?

4.) To play the lottery you have to select 6 out of 59 numbers. How many different kinds of lottery tickets can you purchase?

5.) Use the Binomial Theorem and Pascal’s triangle to write out the product $(x+y)^6$. (Your answer should have a similar form to the example that was done before the exercises.)

6.) Use the Binomial Theorem and Pascal’s triangle to write out the product $(x+y)^7$.

7.) Find $(x+2)^3$ using the Binomial Theorem.

8.) Find $(2x-1)^4$ using the Binomial Theorem.