Sums & Series

Suppose $a_1, a_2, ...$ is a sequence.

Sometimes we'll want to sum the first k numbers (also known as *terms*) that appear in a sequence. A shorter way to write $a_1 + a_2 + a_3 + \cdots + a_k$ is as

$$\sum_{i=1}^{k} a_i$$

There are four rules that are important to know when using $\sum$. They are listed below. In all of the rules, $a_1, a_2, a_3, ...$ and $b_1, b_2, b_3, ...$ are sequences and $c \in \mathbb{R}$.

Rule 1. $$c\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} ca_i$$

Rule #1 is the distributive law. It's another way of writing the equation

$$c(a_1 + a_2 + \cdots + a_k) = ca_1 + ca_2 + \cdots + ca_k$$

Rule 2. $$\sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i = \sum_{i=1}^{k}(a_i + b_i)$$

This rule is essentially another form of the commutative law for addition. It's another way of writing that

$$a_1 + a_2 + \cdots + a_k + b_1 + b_2 + \cdots + b_k = a_1 + b_1 + a_2 + b_2 + \cdots + a_k + b_k$$

Rule 3. $$\sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i = \sum_{i=1}^{k}(a_i - b_i)$$

Rule #3 is a combination of the first two rules. To see that, remember that $-b_i = (-1)b_i$, so we can use Rule #1 (with $c = -1$) followed by Rule #2 to derive Rule #3, as is shown below:

$$\begin{aligned}\sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i &= \sum_{i=1}^{k} a_i + \sum_{i=1}^{k} -b_i \\ &= \sum_{i=1}^{k} (a_i + (-b_i)) \\ &= \sum_{i=1}^{k} (a_i - b_i)\end{aligned}$$

Rule 4. $\displaystyle\sum_{i=1}^{k} c = kc$

The fourth rule can be a little tricky. The number c does not depend on i — it's a constant — so $\sum_{i=1}^{k} c$ is taken to mean that you should add the first k terms in the sequence $c, c, c, c, ...$ That is to say that

$$\sum_{i=1}^{k} c = c + c + \cdots + c = kc$$

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Sum of first k terms in an arithmetic sequence

If $a_1, a_2, a_3, ...$ is an arithmetic sequence, then $a_{n+1} = a_n + d$ for some $d \in \mathbb{R}$. We want to show that

$$\sum_{i=1}^{k} a_i = \frac{k}{2}(a_1 + a_n)$$

To show this, let's write the sum in question in two different ways: front-to-back, and back-to-front. That is,

$$\sum_{i=1}^{k} a_i = a_1 \qquad\qquad + (a_1 + d) + (a_1 + 2d) + \cdots + (a_k - 2d) + (a_k - d) + a_k$$

and

$$\sum_{i=1}^{k} a_i = a_k \qquad\qquad + (a_k - d) + (a_k - 2d) + \cdots + (a_1 + 2d) + (a_1 + d) + a_1$$

Add the two equations above "top-to-bottom" to get

$$2\sum_{i=1}^{k} a_i = [a_1 + a_k] + [a_1 + a_k] + [a_1 + a_k] + \cdots + [a_1 + a_k] + [a_1 + a_k] + [a_1 + a_k]$$

Count and check that there are exactly k of the $[a_1 + a_k]$ terms in the line above being added. Thus,

$$2\sum_{i=1}^{k} a_i = k[a_1 + a_k]$$

which is equivalent to what we were trying to show:

$$\sum_{i=1}^{k} a_i = \frac{k}{2}(a_1 + a_k)$$

Example. What is the sum of the first 63 terms of the sequence $-1, 2, 5, 8, ...$?

The sequence above is arithmetic, because each term in the sequence is 3 plus the term before it, so $d = 3$. The first term of the sequence is -1, so $a_1 = -1$. Our formula $a_n = a_1 + (n-1)d$ tells us that $a_{63} = -1 + (62)3 = 185$. Therefore,

$$\sum_{i=1}^{63} a_i = \frac{63}{2}(-1 + 185) = \frac{63}{2}(184) = 5,796$$

Example. The sum of the first 201 terms of the sequence $10, 17, 24, 31, ...$ equals $\frac{201}{2}(10 + 1410) = \frac{201}{2}(1420) = 142,710$.

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Geometric series

It usually doesn't make any sense at all to talk about adding infinitely many numbers. But if $a_1, a_2, a_3, ...$ is a geometric sequence where $a_{n+1} = ra_n$ and $-1 < r < 1$, then we can make sense of adding all of the terms of the sequence together. (We'll see why later in the semester.)

We will use the symbols

$$\sum_{i=1}^{\infty} a_i$$

to represent adding all of the numbers in the sequence $a_1, a_2, a_3, ...$, and we call this infinite "sum" a *series*.

For the moment, let $S = a_1 + a_2 + a_3 + a_4 + \cdots$. Then

$$S = a_1 + ra_1 + r^2a_1 + r^3a_1 + \cdots$$

and using the distributive law we have

$$rS = \qquad ra_1 + r^2a_1 + r^3a_1 + \cdots$$

Thus, $S - rS = a_1$. Since the distributive law tells us that $S - rS = S(1-r)$, we have $S(1-r) = a_1$, or in other words, $S = \frac{a_1}{1-r}$. We have shown that

$$\sum_{i=1}^{\infty} a_i = \frac{a_1}{1-r}$$

Examples.

- The sum of the terms in the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$ equals 2. We know the sequence is geometric, follows the rule $a_{n+1} = \frac{1}{2}a_n$, and that the first term in the sequence equals 1. Thus

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

- The sum of the terms in the sequence $5, \frac{5}{3}, \frac{5}{9}, \frac{5}{27}, ...$ equals

$$\frac{5}{1-\frac{1}{3}} = \frac{5}{\frac{2}{3}} = \frac{15}{2}$$

Caution. If $a_1, a_2, a_3, ...$ isn't geometric, or if it is but either $r \geq 1$ or $r \leq -1$, then

$$\sum_{i=1}^{\infty} a_i$$

probably doesn't make sense.

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Exercises

1.) If the sum of the first 3976 terms of the sequence $a_1, a_2, a_3, \ldots$ equals 114, then what is the sum of the first 3976 terms of the sequence $\frac{3}{2}a_1, \frac{3}{2}a_2, \frac{3}{2}a_3, \ldots$?

Determine what each of the following eight series equals.

2.) $\displaystyle\sum_{i=1}^{50} 3$

3.) $\displaystyle\sum_{i=1}^{100} 49$

4.) $\displaystyle\sum_{i=1}^{78} (-2)$

5.) $\displaystyle\sum_{i=1}^{40} i$

6.) $\displaystyle\sum_{i=1}^{100} i$

7.) $\displaystyle\sum_{i=1}^{900} i$

8.) $\displaystyle\sum_{i=1}^{5} (2i-1)$

9.) $\sum_{i=1}^{4}(i^2 - 2)$

10.) What is the sum of the first 701 terms of the sequence $-5, -1, 3, 7, ...$?

11.) What is the sum of the first 53 terms of the sequence $140, 137, 134, 131, ...$?

12.) What is the sum of the first 100 terms of the sequence $4, 9, 14, 19, ...$?

13.) What is the sum of the first 80 terms of the sequence $53, 54, 55, 56, ...$?

14.) Sum all of the terms of the geometric sequence $20, 5, \frac{5}{4}, \frac{5}{16}, ...$

15.) Sum all of the terms of the geometric sequence $120, 90, \frac{135}{2}, \frac{405}{8}, ...$

16.) Sum all of the terms of the geometric sequence $7, \frac{14}{3}, \frac{28}{9}, \frac{56}{27}, ...$

17.) Sum all of the terms of the geometric sequence $25, 15, 9, \frac{27}{5}, ...$

18.) Sum all of the terms of the geometric sequence $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, ...$

19.) Suppose that you expect to pay \$400 for gas for your car next year, and that each year after that you plan your yearly gas expenditures will increase by \$20. How much will you spend on gas in the next 8 years?

20.) Suppose you are entertaining two different job offers. Job A has a starting salary of \$20,000 and assures you of a raise of \$1,000 per year. Job B offers you a starting salary of \$23,000, with a yearly raise of \$725. Which job will pay you more over the first ten years? How much more?

21.) An oil well currently produces 5 million gallons of oil per year, but the well is drying up, and each year it will produce 60% of what it did the year before. How much oil can be produced from the well before it is completely dry?