

Sequences

A *sequence* is an infinite list of numbers.

Sequences are written in the form

$$a_1, a_2, a_3, a_4, \dots$$

where $a_1 \in \mathbb{R}$, and $a_2 \in \mathbb{R}$, and $a_3 \in \mathbb{R}$, and $a_4 \in \mathbb{R}$, and so on.

A shorter way to write what's above is to say that a sequence is an infinite list

$$a_1, a_2, a_3, a_4, \dots \text{ with } a_n \in \mathbb{R} \text{ for every } n \in \mathbb{N}.$$

A sequence is different from a set in that the order the numbers are written is important in a sequence. For example, $2, 3, 4, 4, 4, 4, 4, \dots$ is the sequence where $a_1 = 2$, $a_2 = 3$, and $a_n = 4$ if $n \geq 3$. This is a different sequence than $3, 2, 4, 4, 4, 4, 4, \dots$

$1, 2, 1, 2, 1, 2, 1, 2, 1, 2, \dots$ and $2, 1, 2, 1, 2, 1, 2, 1, 2, 1, \dots$ are also different sequences. The former is the sequence with $a_n = 1$ if n is odd and $a_n = 2$ if n is even. The latter is the sequence with $a_n = 2$ if n is odd and $a_n = 1$ if n is even.

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Arithmetic sequences

An *arithmetic sequence* is a sequence a_1, a_2, a_3, \dots where there is some number $d \in \mathbb{R}$ such that

$$\frac{a_{n+1} = a_n + d}{}$$

for every $n \in \mathbb{N}$.

Examples.

- If $a_1 = -10$ and $a_{n+1} = a_n + 5$, then the sequence a_1, a_2, a_3, \dots is $-10, -5, 0, 5, 10, 15, 20, 25, \dots$

- Each number in the sequence $3, 13, 23, 33, 43, 53$ is 10 more than the term directly preceding it. Therefore, it is an arithmetic sequence with

$d = 10$. If you're already convinced of this, that's all you need to do. You don't have to check it any further.

If you're not convinced of that, let's check that it's true: The sequence starts with 3, so $a_1 = 3$. The second number in the sequence is 13, so $a_2 = 13$.

The definition of an arithmetic sequence states that $a_2 = a_1 + d$, and we're checking that $d = 10$ works. That means that we have to check that

$$13 = 3 + 10$$

and of course it does, so our formula that $a_{n+1} = a_n + 10$ works when $n = 1$.

Also, $a_3 = 23$ and $a_2 + 10 = 13 + 10 = 23$, so $a_{n+1} = a_n + 10$ when $n = 2$.

Notice too that $a_4 = 33$ and $a_3 + 10 = 23 + 10 = 33$, so $a_{n+1} = a_n + 10$ when $n = 3$.

Similarly, $a_5 = 43$ and $a_4 + 10 = 33 + 10 = 43$, so $a_{n+1} = a_n + 10$ when $n = 4$.

Finally, $a_6 = 53$ and $a_5 + 10 = 43 + 10 = 53$, so $a_{n+1} = a_n + 10$ when $n = 5$.

We have now completed our check that 3, 13, 23, 33, 43, 53 is an arithmetic sequence, since $a_{n+1} = a_n + 10$.

(We really haven't checked that $a_{n+1} = a_n + 10$ for every $n \in \mathbb{N}$, but we've checked it for every n that we possibly could have checked it for, given that we are only told the first six numbers of the sequence. If ever someone writes the first few terms of a sequence, and those first few terms follow an arithmetic pattern, they mean to imply that that pattern will continue forever. Thus, you only ever have to check the numbers of the sequence that were given explicitly to determine if the entire sequence is arithmetic or not.)

• 6, 3, 0, -3, -6, -9, -12, ... is an arithmetic sequence because each number in the sequence is 3 less than the number directly preceding it. That is, $a_{n+1} = a_n - 3$, which is the same as $a_{n+1} = a_n + (-3)$. So in this example, $d = -3$.

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Geometric sequences

A *geometric sequence* is a sequence a_1, a_2, a_3, \dots where there is some number $r \in \mathbb{R}$ such that

$$\frac{a_{n+1} = ra_n}{}$$

for every $n \in \mathbb{N}$.

Examples.

- If $a_1 = 4$ and $a_{n+1} = 3a_n$, then the sequence a_1, a_2, a_3, \dots is
4, 12, 36, 108, ...

- Each number in the sequence 8, 32, 128, 512, ... is exactly 4 times the number directly preceding it. To check that, notice that

$$\begin{aligned} 32 &= 4(8), \\ 128 &= 4(32), \text{ and} \\ 512 &= 4(128). \end{aligned}$$

Since each number is 4 times the number before it, 8, 32, 128, 512, ... is a geometric series with $r = 4$. That is, $a_{n+1} = 4a_n$.

- An important kind of geometric sequence is one where $a_{n+1} = ra_n$ for a number r with $0 < r < 1$. Then each number in the sequence will be smaller than the ones that came before it. For instance, $a_1 = 8$ and $r = \frac{1}{2}$ describes the geometric sequence that starts with 8, and such that every number is half of the number that came before it: 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$...

- The geometric sequence with $a_1 = 5$ and $r = -2$ is 5, -10, 20, -40, 80, ...

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Predicting numbers in an arithmetic sequence

Let's suppose that a_1, a_2, a_3, \dots is an arithmetic sequence. Then for some $d \in \mathbb{R}$, the equation $a_{n+1} = a_n + d$ holds for every $n \in \mathbb{N}$. If we let $n = 1$ then

$$a_2 = a_1 + d$$

If $n = 2$ then

$$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d$$

If $n = 4$ then

$$a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d$$

If $n = 4$ then

$$a_5 = a_4 + d = (a_1 + 3d) + d = a_1 + 4d$$

We could continue this forever, but there's no need to because there is a pattern emerging. That pattern is

$$a_n = a_1 + (n - 1)d$$

Examples.

- What is the 201st number in the sequence $-10, -5, 0, 5, 10, 15, 20, 25, \dots$?

We saw before that $a_1 = -10$ and $a_{n+1} = a_n + 5$. Hence,

$$a_{201} = a_1 + (201 - 1)5 = a_1 + (200)5 = -10 + 1000 = 990$$

- What is the 27th number in the sequence $3, 13, 23, 33, 43, 53, \dots$?

Because $a_1 = 3$ and $a_{n+1} = a_n + 10$,

$$a_{27} = a_1 + (27 - 1)10 = 3 + (26)10 = 263$$

- The 14th number in the sequence $6, 3, 0, -3, -6, -9, -12, \dots$ is

$$6 + 13(-3) = -33$$

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Predicting numbers in a geometric sequence

If a_1, a_2, a_3, \dots is a geometric sequence then there is an $r \in \mathbb{R}$ such that $a_{n+1} = ra_n$ for all $n \in \mathbb{N}$. Using this formula we see that

$$\begin{aligned} a_2 &= ra_1 \\ a_3 &= ra_2 = r(ra_1) = r^2a_1 \\ a_4 &= ra_3 = r(r^2a_1) = r^3a_1 \\ a_5 &= ra_4 = r(r^3a_1) = r^4a_1 \\ &\vdots \\ a_n &= r^{n-1}a_1 \end{aligned}$$

Examples.

- What is the 7th number in the sequence 4, 12, 36, 108, ...?

$a_1 = 4$ and $a_{n+1} = 3a_n$, so

$$a_7 = 3^{(7-1)} a_1 = 3^6 4 = (729)4 = 2916$$

- What is the 14th number in the sequence 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$...?

$$a_{14} = \left(\frac{1}{2}\right)^{13} 8 = \frac{1}{1024}$$

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Sequences as functions

Let's go back to the definition of a sequence. A fancier way to define a sequence is to say that it is a function whose domain is \mathbb{N} and whose target is \mathbb{R} , because a sequence assigns to every natural number n a single real number, namely a_n .

Like many functions, sequences are sometimes described using explicit formulas.

Examples.

- Suppose that a_1, a_2, a_3, \dots is a sequence that is defined by the formula $a_n = 25n^2$. Then

$$a_{10} = (25)10^2 = (25)100 = 2500$$

- If a sequence is defined by the formula $b_n = \frac{n+13}{n+1}$ then

$$b_7 = \frac{7+13}{7+1} = \frac{20}{8} = \frac{5}{2}$$

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Exercises

Decide whether the following six sequences are either arithmetic, geometric, or neither.

1.) $2, 7, 14, 28, \dots$

2.) $-11, -7, -3, 1, 5, \dots$

3.) $3, -3, 3, -3, 3, \dots$

4.) $2, 3, 4, 5, 6, \dots$

5.) $1, 6, 12, 24, \dots$

6.) $1000, 100, 10, 1, \frac{1}{10}, \dots$

In the next four problems you are given an arithmetic sequence. For each one, what is a_1 , and what is the number d such that $a_{n+1} = a_n + d$?

7.) $-1, 4, 9, 14, \dots$

8.) $2, -10, -22, -34, \dots$

9.) $17, 15, 13, 11, \dots$

10.) $3, 7, 11, 15, \dots$

In the next four problems you are given a geometric sequence. For each one, what is a_1 , and what is the number r such that $a_{n+1} = ra_n$?

11.) $15, 5, \frac{5}{3}, \frac{5}{9}, \dots$

12.) $2, 6, 18, 54, \dots$

13.) $-5, -25, -125, -625, \dots$

14.) $4, -8, 16, -32, 64, \dots$

The next three problems involve arithmetic sequences.

15.) What is the 301st number in the sequence $10, 16, 22, 28, \dots$?

16.) What is the 4223rd number in the sequence 5, 7, 9, 11, ...?

17.) What is the 5224th number in the sequence 4, 1, -2, -5, ...?

Arithmetic sequences don't grow very fast, so it's a doable problem in most cases to find future numbers in a sequence. For example, if you are asked to find a_{327854} in an arithmetic sequence where $a_{n+1} = a_n + d$, the main computational step is to find $(327853)d$. If you know what d is, you could probably do this by hand. If not, a calculator could certainly do this.

In contrast, it is difficult to find large powers of a number. Powers of numbers greater than 1 tend to be so large that calculators can't display the number on their screen, even if they could calculate them. For example, if you are dealing with a sequence where $a_{n+1} = ra_n$, then to find a_{327854} involves finding r^{327853} . That's a really, really big number, even if r isn't so big. For example if $r = 3$, then $r^{327853} = 3^{327853}$, and hand-held calculators won't be able to help you with that.

So for the next three problems, you are asked to find numbers in the sequence, but because they are geometric sequences, you are only asked to find some of the first few numbers in the sequences.

18.) What is the 7th number in the sequence 54, 18, 6, 2, ...?

19.) What is the 6th number in the sequence -11, 22, -44, 88, ...?

20.) What is the 8th number in the sequence 2000, 200, 20, 2, ...?

The last three problems deal with sequences that are defined using explicit equations.

21.) Suppose the sequence a_1, a_2, a_3, \dots is defined by $a_n = 3n - 4$. Find a_{200} .

22.) Let b_1, b_2, b_3, \dots be the sequence where $b_n = \frac{n^2+1}{n}$. Find b_{20} .

23.) If c_1, c_2, c_3, \dots is the sequence $c_n = (3 - n)(n + 2)$, what is c_8 ?