Determinants & Inverse Matrices

The *determinant* of the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the number $ad - cb$.

The above sentence is abbreviated as

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

Example.

$$\det \begin{pmatrix} 4 & -2 \\ 1 & -3 \end{pmatrix} = 4(-3) - 1(-2) = -12 + 2 = -10$$

The determinant of a 3×3 matrix can be found using the formula

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

Example.

$$\det \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 1 & 0 & 1 \end{pmatrix} = 2 \det \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix} - (-1) \det \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix} + 0 \det \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$$

$$= 2[3 \cdot 1 - 0(-2)] + [0 \cdot 1 - 1(-2)] + 0$$

$$= 2 \cdot 3 + 2$$

$$= 8$$

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Determinants and inverses

A matrix has an inverse exactly when its determinant is not equal to 0.

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2×2 inverses

Suppose that the determinant of the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

does not equal 0. Then the matrix has an inverse, and it can be found using the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Notice that in the above formula we are allowed to divide by the determinant since we are assuming that it's not 0.

Example. To find

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}^{-1}$$

first check that

$$\det \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = 3 \cdot 2 - 1 \cdot 5 = 1$$

Then

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{1} \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$$

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3×3 inverses

There is a way to find an inverse of a 3×3 matrix – or for that matter, an $n \times n$ matrix – whose determinant is not 0, but it isn't quite as simple as finding the inverse of a 2×2 matrix. You can learn how to do it if you take a linear algebra course. You could also find websites that will invert matrices for you, and some calculators can find the inverses of matrices as long as the matrices are not too large.

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Exercises

For #1-6, compute the given determinant.

1.)
$$\det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

2.)
$$\det \begin{pmatrix} 1 & \pi \\ 0 & 1 \end{pmatrix}$$

3.)
$$\det \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}$$

4.)
$$\det \begin{pmatrix} 3 & 0 & 0 \\ 107 & 1 & 0 \\ \sqrt{2} & 2 & 6 \end{pmatrix}$$

5.)
$$\det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$$

6.)
$$\det \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 4 \end{pmatrix}$$

7.) Which of the six matrices in the previous problems have an inverse?

8.) What is the inverse of
$$\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} \ ?$$

9.) What is the inverse of
$$\begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix} ?$$

10.) What is the inverse of
$$\begin{pmatrix} 4 & 2 \\ -1 & 3 \end{pmatrix} ?$$

11.) Are the following pair of matrices inverses of each other?

$$\begin{pmatrix} 4 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 3 \\ -1 & 0 & 2 \end{pmatrix}$$

12.) Are the following pair of matrices inverses of each other?

$$\begin{pmatrix} -1 & 1 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$