

# Linear Equations in Two Variables

In this chapter, we'll use the geometry of lines to help us solve equations.

## Linear equations in two variables.

If  $a$ ,  $b$ , and  $r$  are real numbers (and if  $a$  and  $b$  are not both equal to 0) then  $ax + by = r$  is called a *linear equation in two variables*. (The “two variables” are the  $x$  and the  $y$ .)

The numbers  $a$  and  $b$  are called the *coefficients* of the equation  $ax + by = r$ . The number  $r$  is called the *constant* of the equation  $ax + by = r$ .

**Examples.**  $10x - 3y = 5$  and  $-2x - 4y = 7$  are linear equations in two variables.

## Solutions to equations.

A *solution* to a linear equation in two variables  $ax + by = r$  is a specific point in  $\mathbb{R}^2$  such that when the  $x$ -coordinate of the point is multiplied by  $a$ , and the  $y$ -coordinate of the point is multiplied by  $b$ , and those two numbers are added together, the answer equals  $r$ . (There are always infinitely many solutions to a linear equation in two variables.)

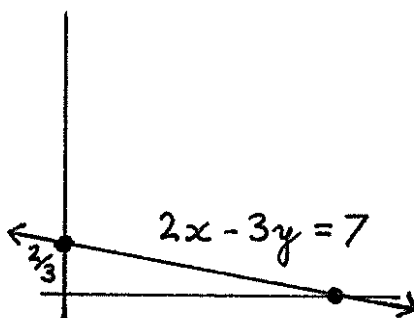
**Example.** Let's look at the equation  $2x - 3y = 7$ .

Notice that  $x = 5$  and  $y = 1$  is a point in  $\mathbb{R}^2$  that is a solution to this equation because we can let  $x = 5$  and  $y = 1$  in the equation  $2x - 3y = 7$  and then we'd have  $2(5) - 3(1) = 10 - 3 = 7$ .

The point  $x = 8$  and  $y = 3$  is also a solution to the equation  $2x - 3y = 7$  since  $2(8) - 3(3) = 16 - 9 = 7$ .

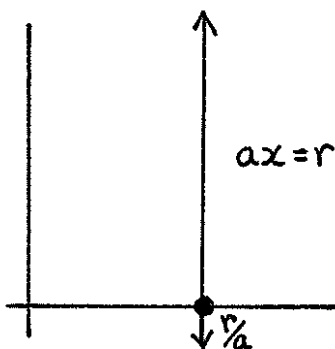
The point  $x = 4$  and  $y = 6$  is not a solution to the equation  $2x - 3y = 7$  because  $2(4) - 3(6) = 8 - 18 = -10$ , and  $-10 \neq 7$ .

To get a geometric interpretation for what the set of solutions of  $2x - 3y = 7$  looks like, we can add  $3y$ , subtract 7, and divide by 3 to rewrite  $2x - 3y = 7$  as  $\frac{2}{3}x - \frac{7}{3} = y$ . This is the equation of a line that has slope  $\frac{2}{3}$  and a  $y$ -intercept of  $-\frac{7}{3}$ . In particular, the set of solutions to  $2x - 3y = 7$  is a straight line. (This is why it's called a linear equation.)

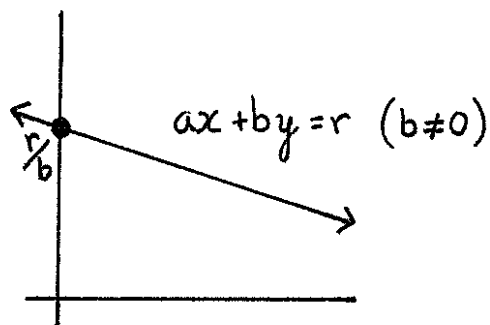


## Linear equations and lines.

If  $b = 0$ , then the linear equation  $ax + by = r$  is the same as  $ax = r$ . Dividing by  $a$  gives  $x = \frac{r}{a}$ , so the solutions to this equation consist of the points on the vertical line whose  $x$ -coordinates equal  $\frac{r}{a}$ .



If  $b \neq 0$ , then the same ideas from the  $2x - 3y = 7$  example that we looked at previously shows that  $ax + by = r$  is the same equation as, just written in a different form from,  $-\frac{a}{b}x + \frac{r}{b} = y$ . This is the equation of a straight line whose slope is  $-\frac{a}{b}$  and whose  $y$ -intercept is  $\frac{r}{b}$ .



## Systems of linear equations.

Rather than asking for the solution set of a single linear equation in two variables, we could take two different linear equations in two variables and ask for all those points that are solutions to *both* of the linear equations.

For example, the point  $x = 4$  and  $y = 1$  is a solution to both of the equations  $x + y = 5$  and  $x - y = 3$ .

If you have more than one linear equation, it's called a *system* of linear equations, so that

$$x + y = 5$$

$$x - y = 3$$

is an example of a system of two linear equations in two variables. There are two equations, and each equation has the same two variables:  $x$  and  $y$ .

A *solution* to a system of equations is a point that is a solution to each of the equations in the system.

**Example.** The point  $x = 3$  and  $y = 2$  is a solution to the system of two linear equations in two variables

$$8x + 7y = 38$$

$$3x - 5y = -1$$

because  $x = 3$  and  $y = 2$  is a solution to  $3x - 5y = -1$  *and* it is a solution to  $8x + 7y = 38$ .

## Unique solutions.

Geometrically, finding a solution to a system of two linear equations in two variables is the same problem as finding a point in  $\mathbb{R}^2$  that lies on each of the straight lines corresponding to the two linear equations.

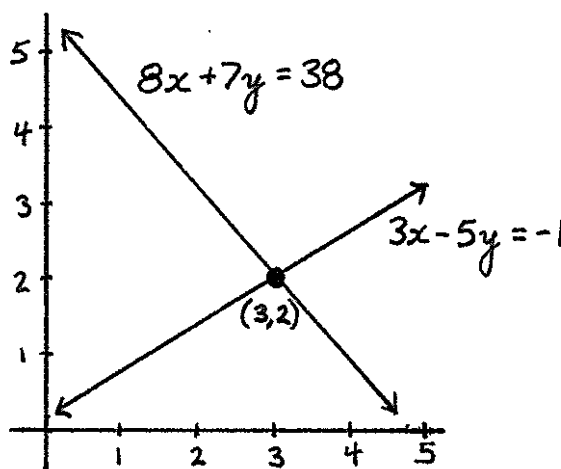
Almost all of the time, two different lines will intersect in a single point, so in these cases, there will only be one point that is a solution to both equations. Such a point is called the *unique* solution for the system of linear equations.

**Example.** Let's take a second look at the system of equations

$$8x + 7y = 38$$

$$3x - 5y = -1$$

The first equation in this system,  $8x + 7y = 38$ , corresponds to a line that has slope  $-\frac{8}{7}$ . The second equation in this system,  $3x - 5y = 3$ , is represented by a line that has slope  $-\frac{3}{-5} = \frac{3}{5}$ . Since the two slopes are not equal, the lines have to intersect in exactly one point. That one point will be the unique solution. As we've seen before that  $x = 3$  and  $y = 2$  is a solution to this system, it must be the unique solution.



**Example.** The system

$$\begin{aligned} 5x + 2y &= 4 \\ -2x + y &= 11 \end{aligned}$$

has a unique solution. It's  $x = -2$  and  $y = 7$ .

It's straightforward to check that  $x = -2$  and  $y = 7$  is a solution to the system. That it's the only solution follows from the fact that the slope of the line  $5x + 2y = 4$  is different from slope of the line  $-2x + y = 11$ . Those two slopes are  $-\frac{5}{2}$  and  $\frac{2}{11}$  respectively.

**No solutions.**

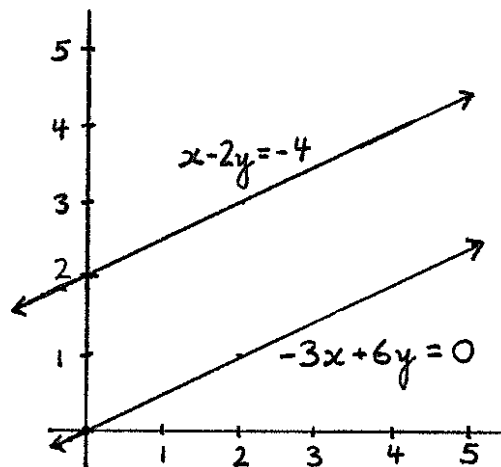
If you reach into a hat and pull out two different linear equations in two variables, it's highly unlikely that the two corresponding lines would have exactly the same slope. But if they did have the same slope, then there

would not be a solution to the system of two linear equations since no point in  $\mathbb{R}^2$  would lie on both of the parallel lines.

**Example.** The system

$$\begin{aligned}x - 2y &= -4 \\ -3x + 6y &= 0\end{aligned}$$

does not have a solution. That's because each of the two lines has the same slope,  $\frac{1}{2}$ , so the lines don't intersect.



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# Exercises

- 1.) What are the coefficients of the equation  $2x - 5y = -23$ ?
- 2.) What is the constant of the equation  $2x - 5y = -23$ ?
- 3.) Is  $x = -4$  and  $y = 3$  a solution to the equation  $2x - 5y = -23$ ?
- 4.) What are the coefficients of the equation  $-7x + 6y = 15$ ?
- 5.) What is the constant of the equation  $-7x + 6y = 15$ ?
- 6.) Is  $x = 3$  and  $y = -10$  a solution to the equation  $-7x + 6y = 15$ ?

- 7.) Is  $x = 1$  and  $y = 0$  a solution to the system

$$\begin{aligned}x + y &= 1 \\2x + 3y &= 3\end{aligned}$$

- 8.) Is  $x = -1$  and  $y = 3$  a solution to the system

$$\begin{aligned}7x + 2y &= -1 \\5x - 3y &= -14\end{aligned}$$

- 9.) What's the slope of the line  $30x - 6y = 3$ ?
- 10.) What's the slope of the line  $-10x + 5y = 4$ ?
- 11.) Is there a unique solution to the system

$$\begin{aligned}30x - 6y &= 3 \\-10x + 5y &= 4\end{aligned}$$

- 12.) What's the slope of the line  $6x + 2y = 4$ ?
- 13.) What's the slope of the line  $15x + 5y = -7$ ?

14.) Is there a unique solution to the system

$$6x + 2y = 4$$

$$15x + 5y = -7$$