# Logarithms

If $a > 1$ or $0 < a < 1$, then the exponential function $f : \mathbb{R} \to (0, \infty)$ defined as $f(x) = a^x$ is one-to-one and onto. That means it has an inverse function.

If either $a > 1$ or $0 < a < 1$, then the inverse of the function $a^x$ is

$$\log_a : (0, \infty) \to \mathbb{R}$$

and it's called a *logarithm* of base $a$.

That $a^x$ and $\log_a(x)$ are inverse functions means that

$$a^{\log_a(x)} = x$$

and

$$\log_a(a^x) = x$$

**Problem.** Find $x$ if $2^x = 15$.

**Solution.** The inverse of an exponential function with base 2 is $\log_2$. That means that we can erase the exponential base 2 from the left side of $2^x = 15$ as long as we apply $\log_2$ to the right side of the equation. That would leave us with $x = \log_2(15)$.

The final answer is $x = \log_2(15)$. You stop there. $\log_2(15)$ is a number. It is a perfectly good number, just like 5, $-7$, or $\sqrt[2]{15}$ are. With some more experience, you will become comfortable with the fact that $\log_2(15)$ cannot be simplified anymore than it already is, just like $\sqrt[2]{15}$ cannot be simplified anymore than it already is. But they are both perfectly good numbers.

**Problem.** Solve for $x$ where $\log_4(x) = 3$.

**Solution.** We can erase $\log_4$ from the left side of the equation by applying its inverse, exponential base 4, to the right side of the equation. That would give us $x = 4^3$. Now $4^3$ can be simplified; it's 64. So the final answer is $x = 64$.

**Problem.** Write $\log_3(81)$ as an integer in standard form.

**Solution.** The trick to solving a problem like this is to rewrite the number being put into the logarithm — in this problem, 81 — as an exponential whose base is the same as the base of the logarithm — in this problem, the base is 3.

Being able to write 81 as an exponential in base 3 will either come from your comfort with exponentials, or from guess-and-check methods. Whether it's immediately obvious to you or not, you can check that $81 = 3^4$. (Notice that $3^4$ is an exponential of base 3.) Therefore, $\log_3(81) = \log_3(3^4)$.

Now we use that exponential base 3 and logarithm base 3 are inverse functions to see that $\log_3(3^4) = 4$.

To summarize this process in one line,

$$\log_3(81) = \log_3(3^4) = 4$$

**Problem.** Write $\log_4(16)$ as an integer in standard form.

**Solution.** This is a logarithm of base 4, so we write 16 as an exponential of base 4: $16 = 4^2$. Then,

$$\log_4(16) = \log_4(4^2) = 2$$

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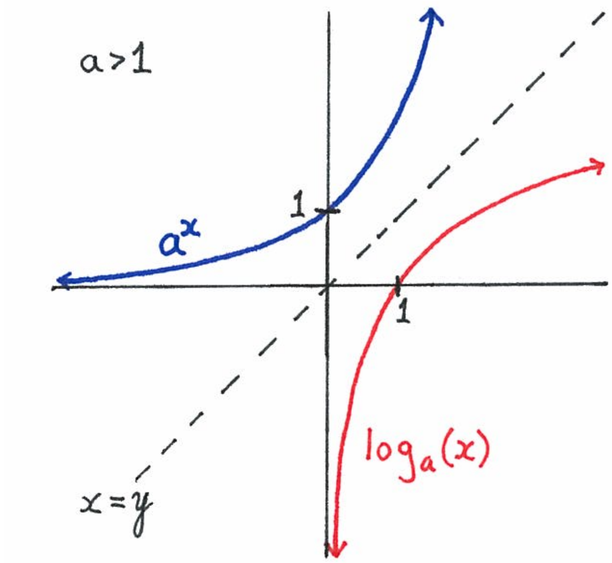
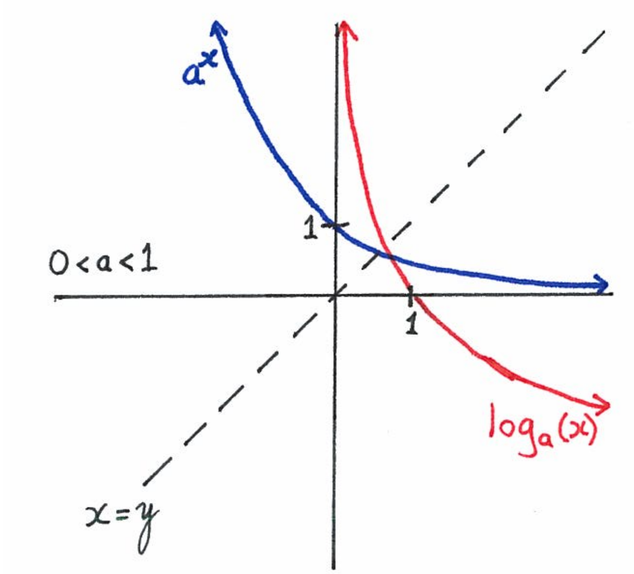
## Graphing logarithms

Recall that if you know the graph of a function, you can find the graph of its inverse function by flipping the graph over the line $x = y$.

Below is the graph of a logarithm of base $a > 1$. Notice that the graph grows taller, but very slowly, as it moves to the right.




Below is the graph of a logarithm when the base is between 0 and 1.

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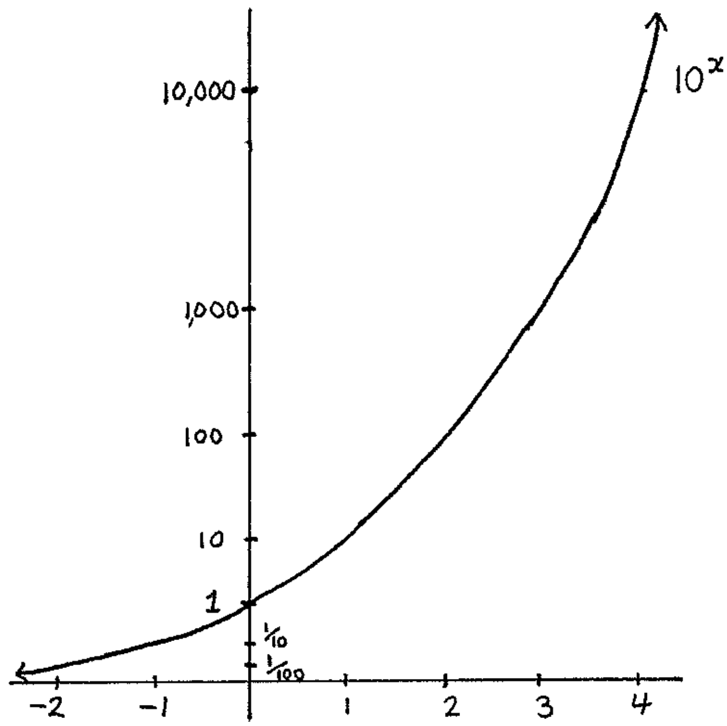
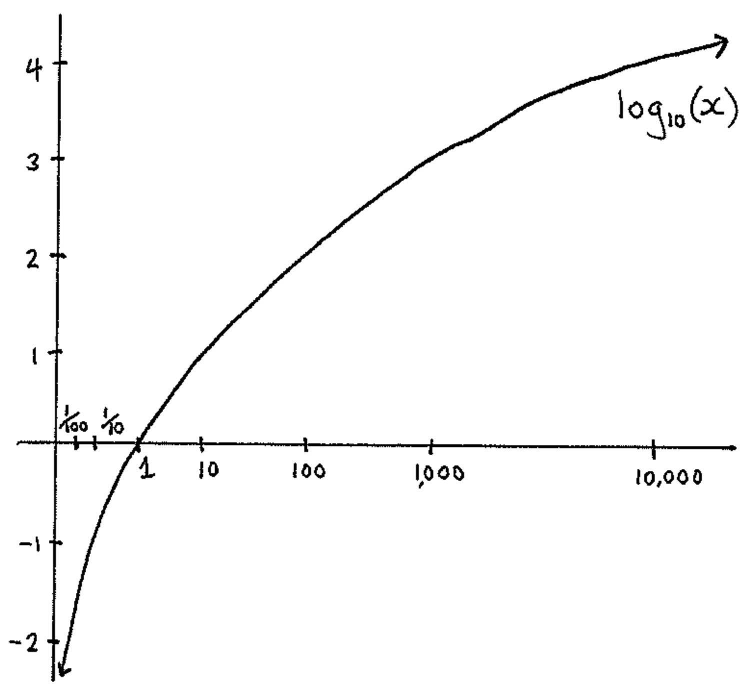
## Two base examples

If $a^x = y$, then $x = \log_a(y)$. Below are some examples in base 10.

| $10^x$ | $\log_{10}(x)$ |
|---|---|
| $10^{-3} = \frac{1}{1{,}000}$ | $-3 = \log_{10}(\frac{1}{1{,}000})$ |
| $10^{-2} = \frac{1}{100}$ | $-2 = \log_{10}(\frac{1}{100})$ |
| $10^{-1} = \frac{1}{10}$ | $-1 = \log_{10}(\frac{1}{10})$ |
| $10^0 = 1$ | $0 = \log_{10}(1)$ |
| $10^1 = 10$ | $1 = \log_{10}(10)$ |
| $10^2 = 100$ | $2 = \log_{10}(100)$ |
| $10^3 = 1,000$ | $3 = \log_{10}(1,000)$ |
| $10^4 = 10,000$ | $4 = \log_{10}(10,000)$ |
| $10^5 = 100,000$ | $5 = \log_{10}(100,000)$ |

Below are the graphs of the functions $10^x$ and $\log_{10}(x)$. The graphs are another way to display the information from the previous chart.

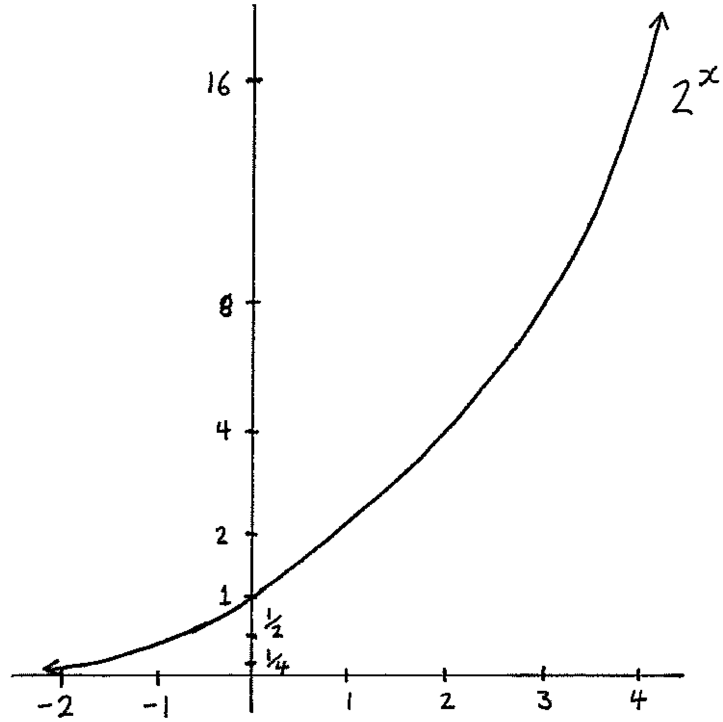


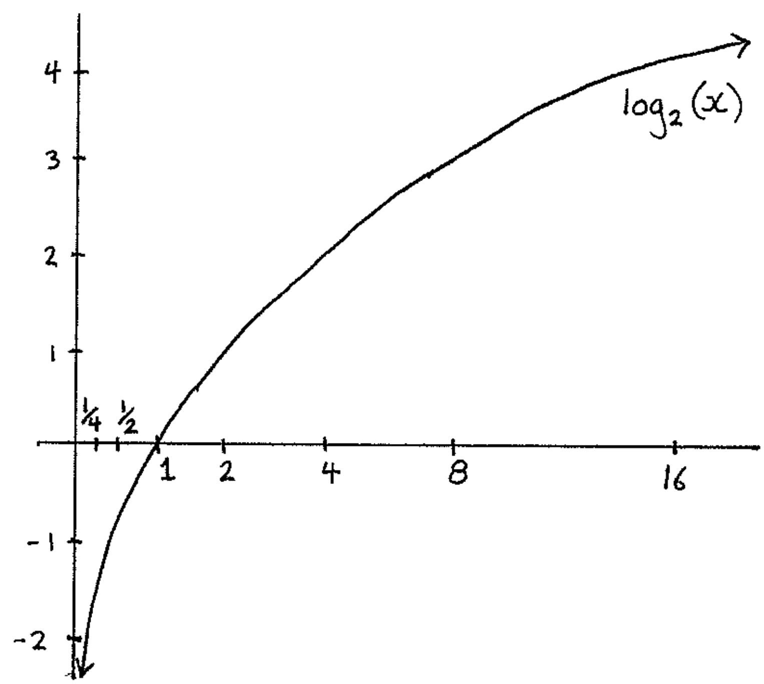



This chart contains examples of exponentials and logarithms in base 2.

| $2^x$ | $\log_2(x)$ |
|---|---|
| $2^{-4} = \frac{1}{16}$ | $-4 = \log_2(\frac{1}{16})$ |
| $2^{-3} = \frac{1}{8}$ | $-3 = \log_2(\frac{1}{8})$ |
| $2^{-2} = \frac{1}{4}$ | $-2 = \log_2(\frac{1}{4})$ |
| $2^{-1} = \frac{1}{2}$ | $-1 = \log_2(\frac{1}{2})$ |
| $2^0 = 1$ | $0 = \log_2(1)$ |
| $2^1 = 2$ | $1 = \log_2(2)$ |
| $2^2 = 4$ | $2 = \log_2(4)$ |
| $2^3 = 8$ | $3 = \log_2(8)$ |
| $2^4 = 16$ | $4 = \log_2(16)$ |
| $2^5 = 32$ | $5 = \log_2(32)$ |
| $2^6 = 64$ | $6 = \log_2(64)$ |

The information from the previous page is used to draw the graphs of $2^x$ and $\log_2(x)$.



## Rules for logarithms

The most important rule for exponential functions is $a^x a^y = a^{x+y}$. Because $\log_a(x)$ is the inverse of $a^x$, it satisfies the "opposite" of this rule:

$$\log_a(z) + \log_a(w) = \log_a(zw)$$

Here's why the above equation is true:

$$\begin{aligned}\log_a(z) + \log_a(w) &= \log_a(a^{\log_a(z)+\log_a(w)})\\ &= \log_a(a^{\log_a(z)}a^{\log_a(w)})\\ &= \log_a(zw)\end{aligned}$$

The next two rules are different versions of the rule above:

$$\log_a(z) - \log_a(w) = \log_a\left(\frac{z}{w}\right)$$

$$\log_a(z^w) = w\log_a(z)$$

Because $a^0 = 1$, it's also true that

$$\log_a(1) = 0$$

## Change of base formula

Let's say that you wanted to know a decimal number that is close to $\log_3(7)$, and you have a calculator that can only compute logarithms in base 10. Your calculator can still help you with $\log_3(7)$ because the change of base formula tells us how to use logarithms in one base to compute logarithms in another base.

The change of base formula is:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

In our example, you could use your calculator to find that 0.845 is a decimal number that is close to $\log_{10}(7)$, and that 0.477 is a decimal number that is close to $\log_{10}(3)$. Then according to the change of base formula

$$\log_3(7) = \frac{\log_{10}(7)}{\log_{10}(3)}$$

is close to the decimal number

$$\frac{0.845}{0.477}$$

which itself is close to 1.771.

We can see why the change of base formula is true. First notice that

$$\log_a(x)\log_b(a) = \log_b(a^{\log_a(x)}) = \log_b(x)$$

The first equal sign above uses the third rule from the section on rules for logarithms. The second equal sign uses that $a^x$ and $\log_a(x)$ are inverse functions.

Now divide the equation above by $\log_b(a)$, and we're left with the change of base formula.

## Base confusion

To a mathematician, $\log(x)$ means $\log_e(x)$. Most calculators use $\log(x)$ to mean $\log_{10}(x)$. Sometimes in computer science, $\log(x)$ means $\log_2(x)$. A lot of people use $\ln(x)$ to mean $\log_e(x)$. ($\ln(x)$ is called the "natural logarithm".)

In this class, we'll never write the expression $\log(x)$ or $\ln(x)$. We'll always be explicit with our bases and write logarithms of base 10 as $\log_{10}(x)$, logarithms of base 2 as $\log_2(x)$, and logarithms of base $e$ as $\log_e(x)$. To be safe, when doing math in the future, always ask what base a logarithm is if it's not clear to you.

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# Exercises

For #1-8, match each of the numbered functions on the left with the lettered function on the right that is its inverse.

| | |
|---|---|
| 1.) $x+7$ | . A.) $x^7$ |
| 2.) $3x$ | B.) $\frac{x}{3}$ |
| 3.) $\sqrt[7]{x}$ | C.) $3^x$ |
| 4.) $7^x$ | D.) $\sqrt[3]{x}$ |
| 5.) $\frac{x}{7}$ | E.) $x-7$ |
| 6.) $\log_3(x)$ | F.) $x+3$ |
| 7.) $x-3$ | G.) $7x$ |
| 8.) $x^3$ | H.) $\log_7(x)$ |

Graph the functions in #9-12.

9.) $\log_{10}(x-3)$

10.) $\log_2(x+5)$

11.) $\log_{\frac{1}{3}}(x)+4$

12.) $-3\log_e(x)$

For #13-21, write the given number as a rational number in standard form, for example, 2, $-3$, $\frac{3}{4}$, and $\frac{-1}{5}$ are rational numbers in standard form. These are the exact same questions, in the same order, as those from #16-24 in the chapter on Exponential Functions. They're just written in the language of logarithms instead.

13.) $\log_4(16)$

14.) $\log_2(8)$

15.) $\log_{10}(10,000)$

16.) $\log_3(9)$

17.) $\log_5(125)$

18.) $\log_{\frac{1}{2}}(16)$

19.) $\log_{\frac{1}{4}}(64)$

20.) $\log_8(\frac{1}{4})$

21.) $\log_{27}(\frac{1}{9})$

For #22-29, decide which is the greatest integer that is less than the given number. For example, if you're given the number $\log_2(9)$ then the answer would be 3. You can see that this is the answer by marking 9 on the $x$-axis of the graph of $\log_2(x)$ that's drawn earlier in this chapter. You can use the graph and the point you marked to see that $\log_2(9)$ is between 3 and 4, so 3 is the greatest of all of the integers that are less than (or below) $\log_2(9)$.

22.) $\log_{10}(15)$

23.) $\log_{10}(950)$

24.) $\log_2(50)$

25.) $\log_2(3)$

26.) $\log_3(18)$

27.) $\log_{10}(\frac{1}{19})$

28.) $\log_2(\frac{1}{10})$

29.) $\log_3(\frac{1}{10})$

In the remaining exercises, use that $\log_a(x)$ and $a^x$ are inverse functions to solve for $x$.

30.) $\log_4(x) = -2$

31.) $\log_6(x) = 2$

32.) $\log_3(x) = -3$

33.) $\log_{\frac{1}{10}}(x) = -5$

34.) $e^x = 17$

35.) $e^x = 53$

36.) $\log_e(x) = 5$

37.) $\log_e(x) = -\frac{1}{3}$