Rational Functions

In this chapter, you'll learn what a rational function is, and you'll learn how to sketch the graph of a rational function.

Rational functions

A *rational function* is a fraction of polynomials. That is, if $p(x)$ and $q(x)$ are polynomials, then

$$\frac{p(x)}{q(x)}$$

is a rational function. The *numerator* is $p(x)$ and the *denominator* is $q(x)$.

Examples.

- $\frac{3(x-5)}{(x-1)}$
- $\frac{1}{x}$
- $\frac{2x^3}{1} = 2x^3$

The last example is both a polynomial and a rational function. In a similar way, any polynomial is a rational function.

In this class, from this point on, most of the rational functions that we'll see will have both their numerators and their denominators completely factored.

We will also only see examples where the numerator and the denominator have no common factors. (If they did have a common factor, we could just cancel them.)

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Implied domains

The implied domain of a rational function is the set of all real numbers *except* for the roots of the denominator. That's because it doesn't make sense to divide by 0.

Example. The implied domain of

$$\frac{-7(x-2)(x^2+1)}{8(x-4)(x-6)}$$

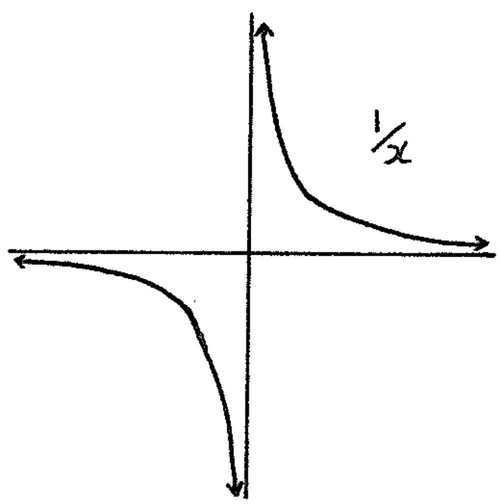
is the set $\mathbb{R} - \{4, 6\}$.

Vertical asymptotes

To graph a rational function, begin by marking every number on the x-axis that is a root of the denominator. (The denominator might not have any roots.)

Draw a vertical dashed line through these points. These vertical lines are called *vertical asymptotes*. The graph of the rational function will "climb up" or "slide down" the sides of a vertical asymptote.

Examples. For the rational function $\frac{1}{x}$, 0 is the only root of the denominator, so the y-axis is the vertical asymptote. Notice that the graph of $\frac{1}{x}$ climbs up the right side of the y-axis and slides down the left side of the y-axis.



The rational function

$$\frac{-7(x-2)(x^2+1)}{8(x-4)(x-6)}$$

has vertical asymptotes at $x = 4$ and at $x = 6$.

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x-intercepts

The x-intercepts of a rational function $\frac{p(x)}{q(x)}$ (if there are any) are the numbers $\alpha \in \mathbb{R}$ where

$$\frac{p(\alpha)}{q(\alpha)} = 0$$

If α is such a number, then we can multiply by $q(\alpha)$ to find that

$$p(\alpha) = 0 \cdot q(\alpha) = 0$$

In other words, α is a root of $p(x)$. Thus, the roots of the numerator are exactly the x-intercepts.

Example. 2 is the only x-intercept of the rational function

$$\frac{-7(x-2)(x^2+1)}{8(x-4)(x-6)}$$

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In between x-intercepts and vertical asymptotes

When graphing a rational polynomial, first mark the vertical asymptotes and the x-intercepts. Then choose a number $c \in \mathbb{R}$ between any consecutive pairs of these marked points on the x-axis and see if the rational function is positive or negative when $x = c$. If it's positive, draw a dot above the x-axis whose first coordinate is c. If it's negative, draw a dot below the x-axis whose first coordinate is c.

Example. Let's look at the function

$$r(x) = \frac{-7(x-2)(x^2+1)}{8(x-4)(x-6)}$$

again. The x-intercept of its graph is at $x = 2$ and it has vertical asymptotes at $x = 4$ and $x = 6$. We need to decide whether $r(x)$ is positive or negative between 2 and 4 on the x-axis, and between 4 and 6 on the x-axis.

Let's start by choosing a number between 2 and 4, say 3. Then

$$r(3) = \frac{-7(3-2)(3^2+1)}{8(3-4)(3-6)}$$

Notice that -7, $(3-4)$, and $(3-6)$ are negative, while 8, $(3-2)$, and (3^2+1) are positive.

If you are multiplying and dividing a collection of numbers that aren't equal to 0, just count how many negative numbers there are. If there is an even number of negatives, the result will be positive. If there is an odd number of negatives, the result will be negative. In the previous paragraph, there are three negative numbers — -7, $(3-4)$, and $(3-6)$ — so $r(3) < 0$.

The number 5 is a number that is in between 4 and 6, and

$$r(5) = \frac{-7(5-2)(3^2+1)}{8(5-4)(5-6)} > 0$$

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Far right and far left

Let ax^n be the leading term of $p(x)$ and let bx^m be the leading term of $q(x)$.

Recall that far to the right and left, $p(x)$ looks like its leading term, ax^n. And far to the right and left, $q(x)$ looks like its leading term, bx^m. It follows that the far right and left portion of the graph of,

$$\frac{p(x)}{q(x)}$$

looks like

$$\frac{ax^n}{bx^m}$$

and this is a function that we know how to graph.

Example. The leading term of $-7(x-2)(x^2+1)$ is $-7x^3$, and the leading term of $8(x-4)(x-6)$ is $8x^2$. Therefore, the graph of

$$r(x) = \frac{-7(x-2)(x^2+1)}{8(x-4)(x-6)}$$

looks like the graph of

$$\frac{-7x^3}{8x^2} = \frac{-7}{8}x$$

on the far left and far right part of its graph.

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Putting the graph together

To graph a rational function

$$\frac{p(x)}{q(x)}$$

mark its vertical asymptotes (if any). Mark its x-intercepts (if any). Determine whether the function is positive or negative in between x-intercepts and vertical asymptotes.

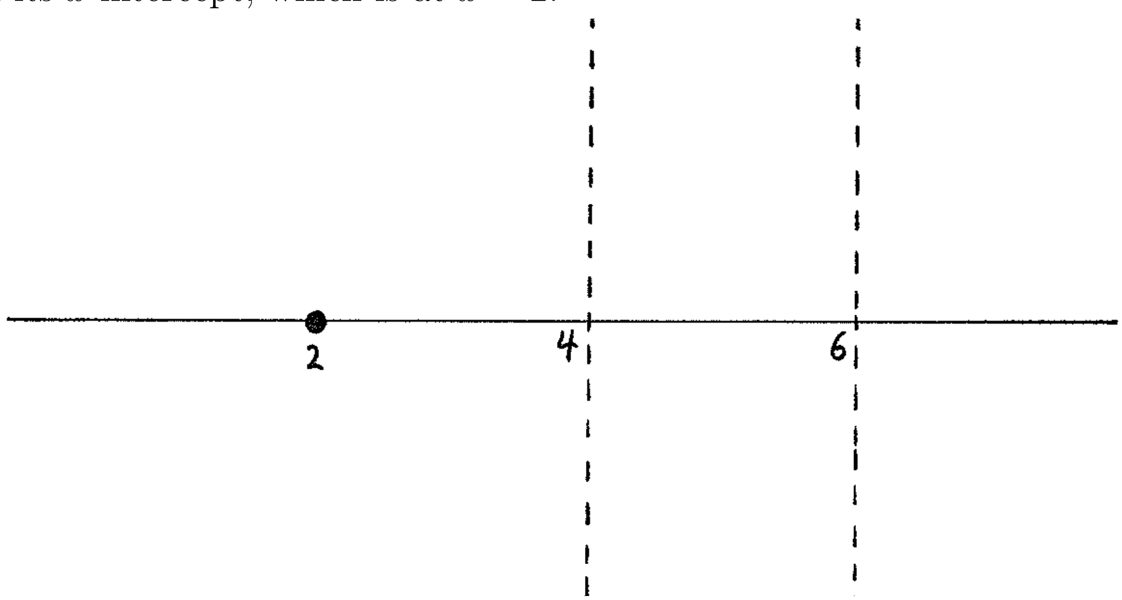
Replace $p(x)$ with its leading term, replace $q(x)$ with its leading term, and then graph the resulting fraction of leading terms to the right and left of everything you've drawn so far in your graph.

Now draw a reasonable looking graph that fits with everything you've drawn so far, remembering that the graph has to climb up or slide down the sides of vertical asymptotes, and that the graph can only touch the x-axis at the x-intercepts that you already marked.

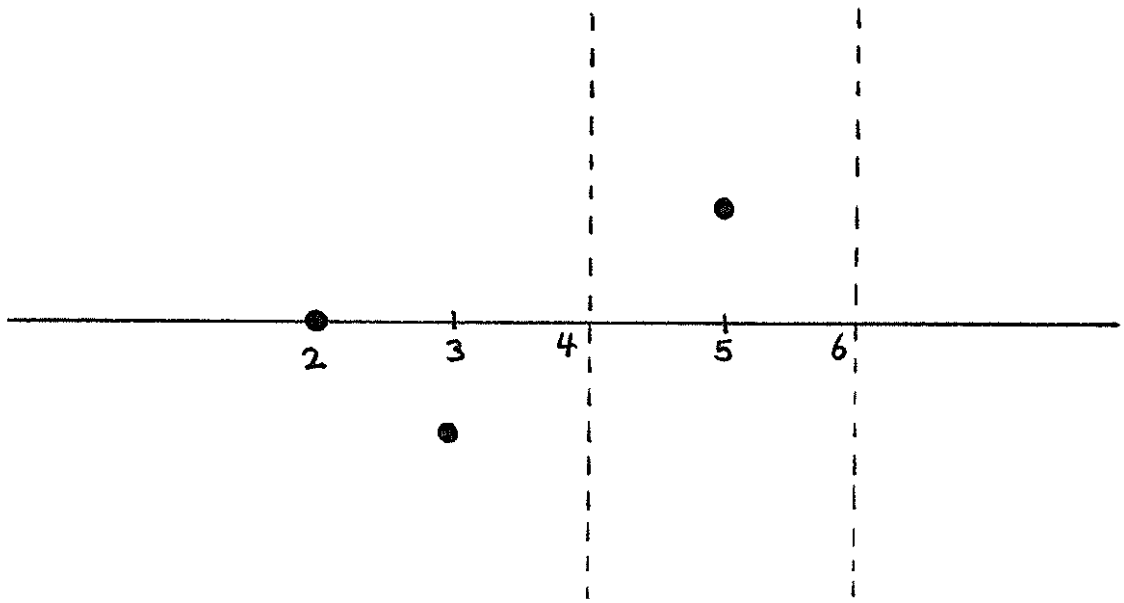
Example. Let's graph

$$r(x) = \frac{-7(x-2)(x^2+1)}{8(x-4)(x-6)}$$

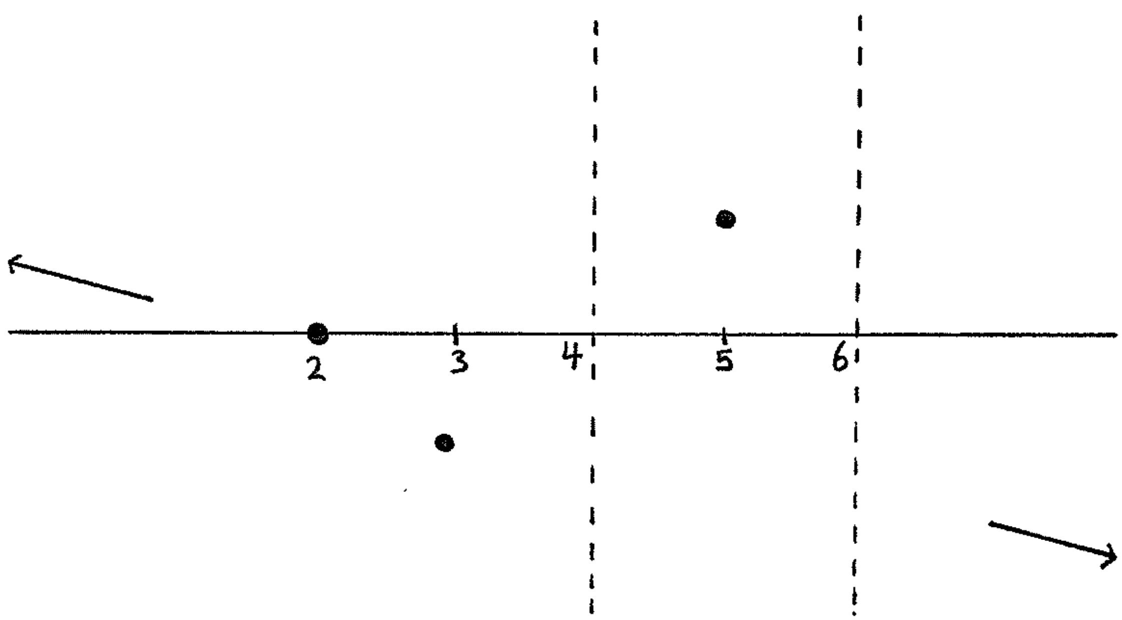
First we mark its its vertical asymptotes, which are at $x = 4$ and $x = 6$, and its x-intercept, which is at $x = 2$.



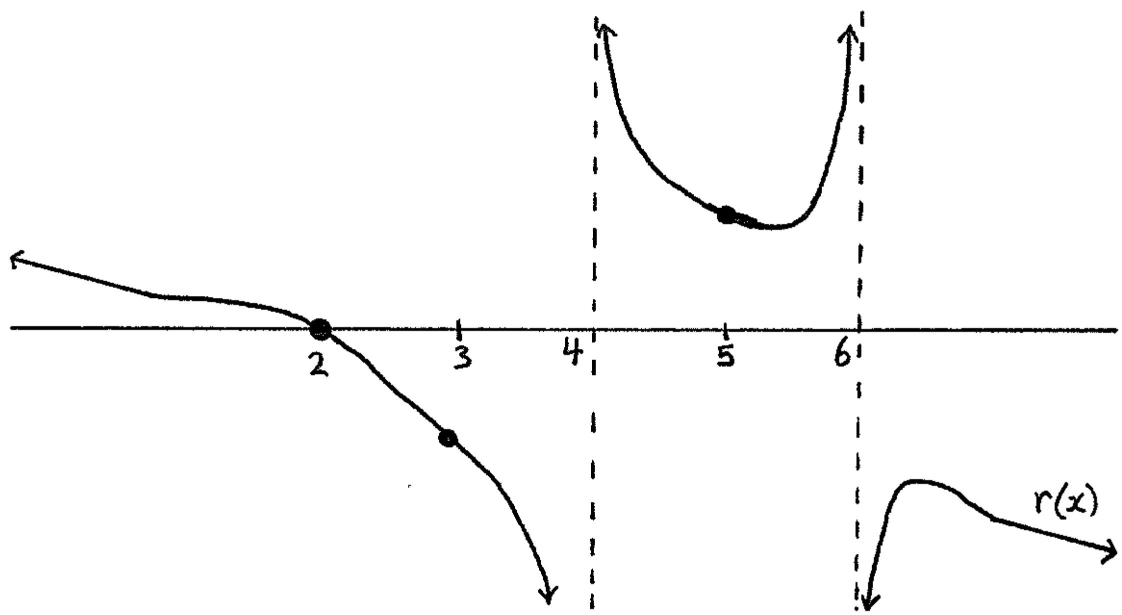
Then we plot points that represent what we had checked earlier for what happens in between consecutive pairs of x-intercepts and vertical asymptotes: that $r(3) < 0$ and $r(5) > 0$.



To the left and right of what we've graphed so far, we draw the graph of $\frac{-7}{8}x$.



Now we connect what we've drawn so far, making sure our graph climbs up or slides down the vertical asymptotes, and that it only touches the x-axis at the previously labelled x-intercept.



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Exercises

For #1-3, use that $4x^2-4=4(x-1)(x+1)$, $x^3-3x^2+4=(x+1)(x-2)^2$, and $2x-4=2(x-2)$ to match each of the three numbered rational functions on the left with its simplified lettered form on the right.

1.) $\frac{4x^2-4}{x^3-3x^2+4}$ A.) $\frac{1}{2}(x+1)(x-2)$

2.) $\frac{x^3-3x^2+4}{2x-4}$ B.) $\frac{4(x-1)}{(x-2)^2}$

3.) $\frac{2x-4}{4x^2-4}$ C.) $\frac{(x-2)}{2(x-1)(x+1)}$

Graph the rational functions given in #4-10. (Their numerators and denominators have been completely factored.)

4.) $\frac{3(x^2+1)}{(x^2+5)}$

5.) $\frac{4(x+1)^2}{2(x+2)(x-2)}$

6.) $\frac{-(x+1)(x^2+1)(x^2+8)}{(x-7)}$

7.) $7(x+2)^3(x-3)^2$

8.) $\frac{(x-4)(x-6)}{(x^2+3)(x^2+4)(x^2+8)}$

9.) $\frac{3(x^2+7)}{5(x-2)^2(x-6)}$

10.) $\frac{2(x+10)^2(x+30)}{-3(x-5)}$

11.) Completely factor the numerator and the denominator of the rational function below, and then graph it.

$$\frac{3x^3-6x^2+x-2}{x^2+3x+2}$$