Graphing Polynomials

In the previous chapter, we learned how to factor a polynomial. In this chapter, we'll use the completely factored form of a polynomial to help us graph it.

The far right and far left of a polyniomial graph

Suppose $p(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$ is a polynomial.

If M is a really big number, then M^n is much bigger than M^{n-1}. (For example, if $M = 1000$ then M^n is one thousand times bigger than M^{n-1}.)

In fact, if M is a really, really big number then M^n is much bigger than $a_{n-1}M^{n-1}$, or $a_{n-2}M^{n-2}$, or $a_{n-3}M^{n-3}$, and so on.

Actually, if M is a really really big number, then a_nM^n is much bigger than the numbers in the previous paragraph, and it even dwarfs their sum:

$$a_{n-1}M^{n-1} + a_{n-2}M^{n-2} + \cdots + a_0$$

That means that for really, really big numbers M in the domain of the polynomial $p(x)$, the size of

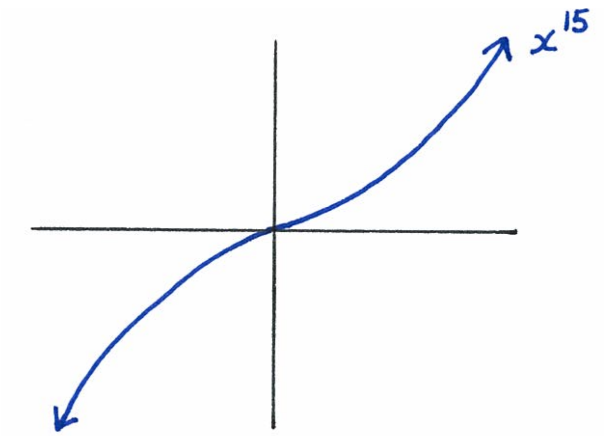
$$p(M) = a_nM^n + a_{n-1}M^{n-1} + a_{n-2}M^{n-2} + \cdots + a_0$$

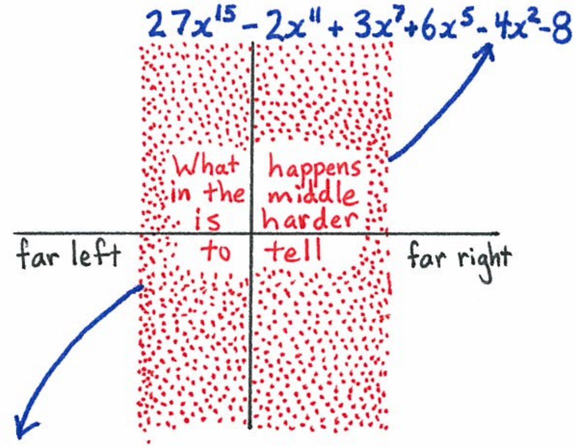
is basically determined by a_nM^n. That's because while the rest of $p(M)$ — which is $a_{n-1}M^{n-1} + a_{n-2}M^{n-2} \cdots + a_0$ — might be large, it is so small in comparison to a_nM^n that it's hard to notice it. In other words, while $p(M)$ does not equal a_nM^n, it's hard to tell the difference between the two in the same way that it would be hard to tell the difference between a bag of $10,000,576$ pennies and a bag of $10,002,073$ pennies; both bags have about 10 million pennies.

The end result is that the graph of $p(x)$ looks an awful lot like the graph of a_nx^n over the part of the x-axis that has the really, really big numbers: the extreme right portion of the x-axis.

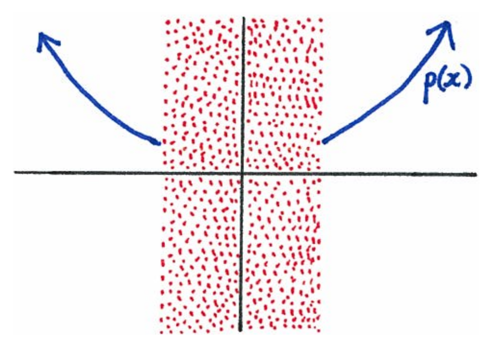
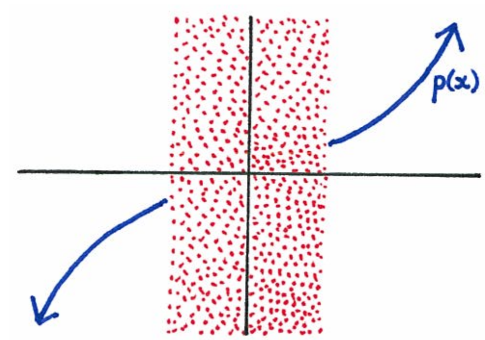
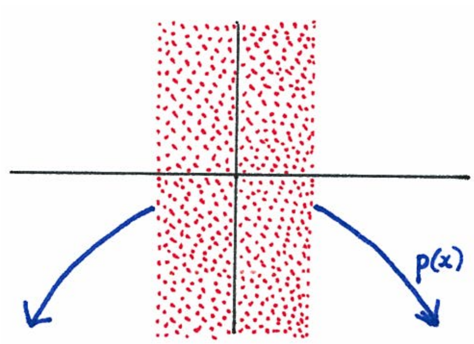
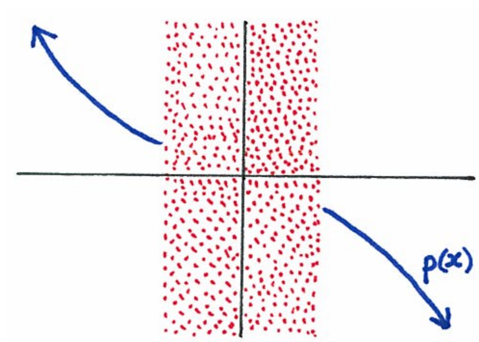
The graph of $p(x)$ also looks an awful lot like the graph of a_nx^n over the extreme left portion of the x-axis.

Example. Over the far left portion of the x-axis, and over the far right portion of the x-axis, the graph of $q(x) = 27x^{15} - 2x^{11} + 3x^7 + 6x^5 - 4x^2 - 8$ basically looks like the graph of its leading term: $27x^{15}$. And the graph of $27x^{15}$ is the graph of x^{15} stretched by 27, which is pretty much the same graph as the graph for x^{15}.





The chart below gives a rough sketch of what the far right and far left portion of the graph of a polynomial $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ looks like (if $n \geq 2$). It just depends on the leading term, a_nx^n. What the middle portion of the graph looks like is harder to say.

	n even	n odd
$a_n > 0$		
$a_n < 0$		

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Steps for graphing a completely factored polynomial $p(x)$.

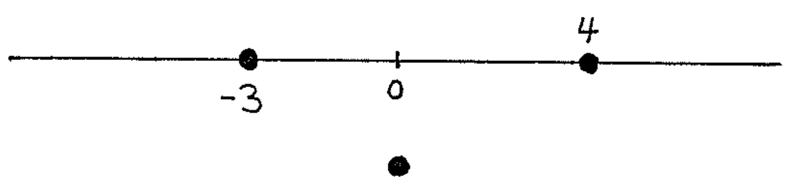
1: The roots of $p(x)$ are the x-intercepts. You can read them off from the monic linear factors of a completely factored polynomial.

The polynomial $p(x) = -5(x+3)(x-4)(x-4)(x^2+2x+6)$ is completely factored. Its roots are -3 and 4.

2: Pick any number on the x-axis between consecutive pairs of x-intercepts.

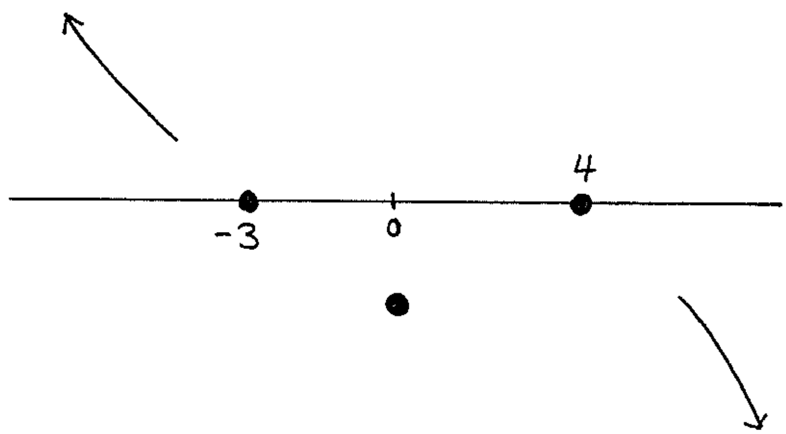
Let's say the number you picked was b. If $p(b) > 0$, but a giant dot directly above the b. Put a giant dot directly below b if $p(b) < 0$.

0 is a number in between -3 and 4, and for $p(x) = -5(x+3)(x-4)(x-4)(x^2+2x+6)$ we have $p(0) = -5(3)(-4)(-4)(6) < 0$, so we can put a giant dot directly below 0.

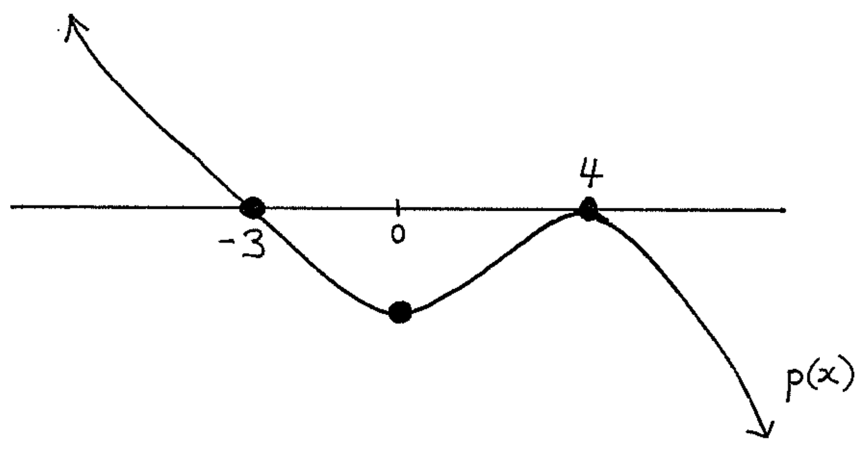


3: The far right and left portion of the graph of $p(x)$ looks like the graph of its leading term. Draw what the graph of the leading term looks like on the far right and left sides of your picture.

The leading term of $p(x) = -5(x+3)(x-4)(x-4)(x^2+2x+6)$ is $-5x^5$, and that looks like the graph of x^5 turned upside down.



4: Draw any type of smooth, curvy, and continuous line that passes through all of the points in $\mathbb{R}^2$ that you labeled from Steps 1 and 2, that does not touch the x-axis at any points not listed in Step 1, and that meets up with the pieces of the graph you drew in Step 3.



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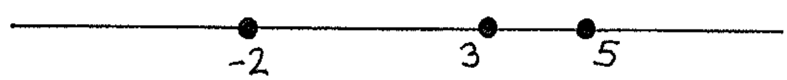
Problem. The polynomial $q(x) = 7(x+2)(x-3)(x-5)(x^2+1)$ is completely factored. Graph it.

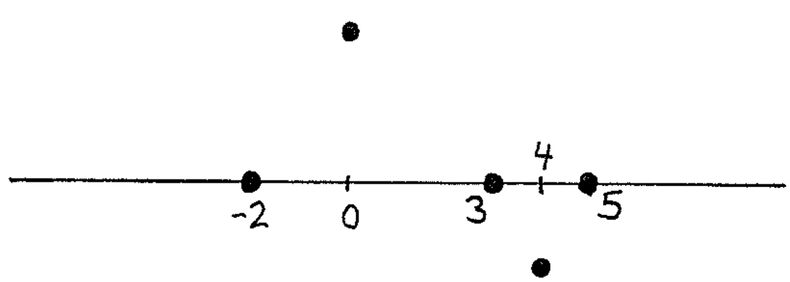
Solution.

1: The monic linear factors of $q(x)$ are $(x+2)$, $(x-3)$, and $(x-5)$. So the roots of $q(x)$ are -2, 3, and 5. Draw these three points on the x-axis.

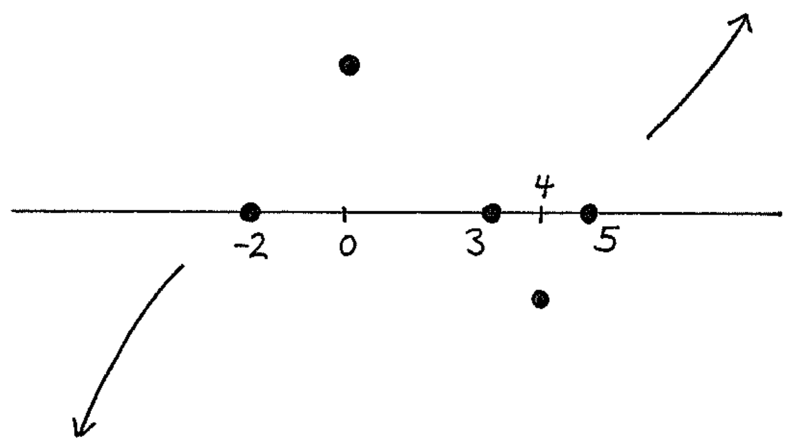


2: Choose any number between -2 and 3, for example, the number 0 is between -2 and 3. Then check to see if $q(0)$ is positive or negative: $q(0) = 7(0+2)(0-3)(0-5)(0^2+1) = 7(2)(-3)(-5)(1)$ is a positive number, so draw a dot above 0.

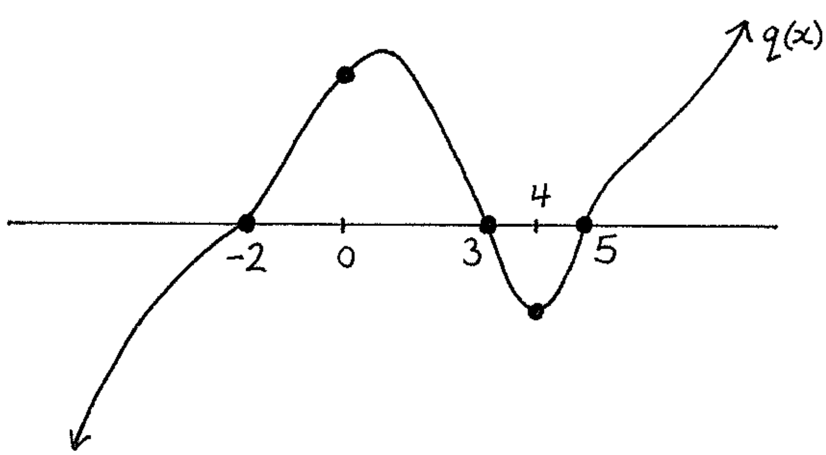
Similarly, choose a point between 3 and 5, say 4. Check that $q(4) = 7(4+2)(4-3)(4-5)(4^2+1) = 7(6)(1)(-1)(17)$ is negative, so we draw a dot below 4.



3: The leading term of $q(x)$ is $7x^5$. Draw the part of $7x^5$ that is to the left of everything you've drawn in your picture so far. Draw the part of the graph of $7x^5$ that is to the right of everything you've drawn so far.



4: Draw the graph of a function that connects everything you've drawn, but make sure it only touches the x-axis at the x-intercepts that you've already labelled. That is more or less what the graph of $q(x)$ looks like. That's our answer.



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Exercises

For #1-6, graph the given completely factored polynomials.

1.) $4(x-3)(x-5)$

2.) $-(x-1)(x^2+x+5)$

3.) $-6(x+4)(x+4)(x-2)(x-3)(x^2+1)(x^2+3)$

4.) $2(x-3)(x-3)(x-3)(x-6)(x-6)(x^2+2x+7)$

5.) $-(x-1)(x-2)(x-2)(x-3)(x^2+x+5)$

6.) $5(x-4)(x^2-2x+4)(x^2+3x+5)(x^2+4)$

For #7-10, first completely factor, and then graph, the given polynomials.

7.) $2x^2+2x-24$

8.) $7x^2-3x+4$

9.) $-x^3+6x^2+7x$

10.) $3x^4-9x^2-12$

11.) How can what we know about the far right and left of the graph of an odd degree polynomial be used to show that any polynomial of odd degree has at least one root?