Factoring Polynomials

Any natural number that is greater than 1 can be factored into a product of prime numbers. For example $20 = (2)(2)(5)$ and $30 = (2)(3)(5)$.

In this chapter we'll learn an analogous way to factor polynomials.

Fundamental Theorem of Algebra

A *monic* polynomial is a polynomial whose leading coefficient equals 1. So $x^4 - 2x^3 + 5x - 7$ is monic, and $x - 2$ is monic, but $3x^2 - 4$ is not monic.

Carl Friedrich Gauss was the boy who discovered a really quick way to see that $1 + 2 + 3 + \cdots + 100 = 5050$.

In 1799, a grown-up Gauss proved the following theorem:

Any polynomial is the product of a real number,
and a collection of monic quadratic polynomials that
do not have roots, and of monic linear polynomials.

This result is called the *Fundamental Theorem of Algebra*. It is one of the most important results in all of mathematics, though from the form it's written in above, it's probably difficult to immediately understand its importance.

The explanation for why this theorem is true is somewhat difficult, and it is beyond the scope of this course. We'll have to accept it on faith.

Examples.

- $4x^2 - 12x + 8$ can be factored into a product of a number, 4, and two monic linear polynomials, $x - 1$ and $x - 2$. That is, $4x^2 - 12x + 8 = 4(x - 1)(x - 2)$.

- $-x^5+2x^4-7x^3+14x^2-10x+20$ can be factored into a product of a number, -1, a monic linear polynomial, $x-2$, and two monic quadratic polynomials that don't have roots, x^2+2 and x^2+5. That is $-x^5+2x^4-7x^3+14x^2-10x+20 = -(x-2)(x^2+2)(x^2+5)$. (We can check the discriminants of x^2+2 and x^2+5 to see that these two quadratics don't have roots.)

- $2x^4-2x^3+14x^2-6x+24 = 2(x^2+3)(x^2-x+4)$. Again, x^2+3 and x^2-x+4 do not have roots.

Notice that in each of the above examples, the real number that appears in the product of polynomials – 4 in the first example, -1 in the second, and 2 in the third – is the same as the leading coefficient for the original polynomial. This always happens, so the Fundamental Theorem of Algebra can be more precisely stated as follows:

If $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$,
then $p(x)$ is the product of the real number a_n,
and a collection of monic quadratic polynomials that
do not have roots, and of monic linear polynomials.

Completely factored

A polynomial is *completely factored* if it is written as a product of a real number (which will be the same number as the leading coefficient of the polynomial), and a collection of monic quadratic polynomials that do not have roots, and of monic linear polynomials.

Looking at the examples above, $4(x-1)(x-2)$ and $-(x-2)(x^2+2)(x^2+5)$ and $2(x^2+3)(x^2-x+4)$ are completely factored.

One reason it's nice to completely factor a polynomial is because if you do, then it's easy to read off what the roots of the polynomial are.

Example. Suppose $p(x) = -2x^5+10x^4+2x^3-38x^2+4x-48$. Written in this form, its difficult to see what the roots of $p(x)$ are. But after being completely factored, $p(x) = -2(x+2)(x-3)(x-4)(x^2+1)$. The roots of this polynomial can be read from the monic linear factors. They are -2, 3, and 4.

(Notice that $p(x) = -2(x+2)(x-3)(x-4)(x^2+1)$ is completely factored because x^2+1 has no roots.)

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Factoring linears

To completely factor a linear polynomial, just factor out its leading coefficient:

$$ax + b = a\Big(x + \frac{b}{a}\Big)$$

For example, to completely factor $2x + 6$, write it as the product $2(x+3)$.

Factoring quadratics

What a completely factored quadratic polynomial looks like will depend on how many roots it has.

0 Roots. If the quadratic polynomial $ax^2 + bx + c$ has 0 roots, then it can be completely factored by factoring out the leading coefficient:

$$ax^2 + bx + c = a\Big(x^2 + \frac{b}{a}x + \frac{c}{a}\Big)$$

(The graphs of ax^2+bx+c and $x^2+\frac{b}{a}x+\frac{c}{a}$ differ by a vertical stretch or shrink that depends on a. A vertical stretch or shrink of a graph won't change the number of x-intercepts, so $x^2+\frac{b}{a}x+\frac{c}{a}$ won't have any roots since ax^2+bx+c doesn't have any roots. Thus, $x^2+\frac{b}{a}x+\frac{c}{a}$ is completely factored.)

Example. The discriminant of $4x^2-2x+2$ equals $(-2)^2-4(4)(2) = 4-32 = -28$, a negative number. Therefore, $4x^2 - 2x + 2$ has no roots, and it is completely factored as $4(x^2 - \frac{1}{2}x + \frac{1}{2})$.

2 Roots. If the quadratic polynomial $ax^2 + bx + c$ has 2 roots, we can name them α_1 and α_2. Roots give linear factors, so we know that $(x - \alpha_1)$

and $(x - \alpha_2)$ are factors of $ax^2 + bx + c$. That means that there is some polynomial $q(x)$ such that

$$ax^2 + bx + c = q(x)(x - \alpha_1)(x - \alpha_2)$$

The degree of $ax^2 + bx + c$ equals 2. Because the sum of the degrees of the factors equals the degree of the product, we know that the degree of $q(x)$ plus the degree of $(x - \alpha_1)$ plus the degree of $(x - \alpha_2)$ equals 2. In other words, the degree of $q(x)$ plus 1 plus 1 equals 2.

Zero is the only number that you can add to $1 + 1$ to get 2, so $q(x)$ must have degree 0, which means that $q(x)$ is just a constant number.

Because the leading term of $ax^2 + bx + c$ – namely ax^2 – is the product of the leading terms of $q(x)$, $(x - \alpha_1)$, and $(x - \alpha_2)$ – namely the number $q(x)$, x, and x – it must be that $q(x) = a$. Therefore,

$$ax^2 + bx + c = a(x - \alpha_1)(x - \alpha_2)$$

Example. The discriminant of $2x^2 + 4x - 2$ equals $4^2 - 4(2)(-2) = 16 + 16 = 32$, a positive number, so there are two roots.

We can use the quadratic formula to find the two roots, but before we do, it's best to simplify the square root of the discriminant: $\sqrt{32} = \sqrt{(4)(4)(2)} = 4\sqrt{2}$.

Now we use the quadratic formula to find that the roots are

$$\frac{-4 + 4\sqrt{2}}{2(2)} = \frac{-4 + 4\sqrt{2}}{4} = -1 + \sqrt{2}$$

and

$$\frac{-4 - 4\sqrt{2}}{2(2)} = \frac{-4 - 4\sqrt{2}}{4} = -1 - \sqrt{2}$$

Therefore, $2x^2 + 4x - 2$ is completely factored as

$$2\Big(x - (-1 + \sqrt{2})\Big)\Big(x - (-1 - \sqrt{2})\Big) = 2(x + 1 - \sqrt{2})(x + 1 + \sqrt{2})$$

1 Root. If $ax^2 + bx + c$ has exactly 1 root (let's call it α_1) then $(x - \alpha_1)$ is a factor of $ax^2 + bx + c$. Hence,

$$ax^2 + bx + c = g(x)(x - \alpha_1)$$

for some polynomial $g(x)$.

Because the degree of a product is the sum of the degrees of the factors, $g(x)$ must be a degree 1 polynomial, and it can be completely factored into something of the form $\lambda(x-\beta)$ where $\lambda, \beta \in \mathbb{R}$. Therefore,

$$ax^2 + bx + c = \lambda(x-\beta)(x-\alpha_1)$$

Notice that β is a root of $\lambda(x-\beta)(x-\alpha_1)$, so β is a root of ax^2+bx+c since they are the same polynomial. But we know that ax^2+bx+c has only one root, namely α_1, so β must equal α_1. That means that

$$ax^2 + bx + c = \lambda(x-\alpha_1)(x-\alpha_1)$$

The leading term of ax^2+bx+c is ax^2. The leading term of $\lambda(x-\alpha_1)(x-\alpha_1)$ is λx^2. Since ax^2+bx+c equals $\lambda(x-\alpha_1)(x-\alpha_1)$, they must have the same leading term. Therefore, $ax^2 = \lambda x^2$. Hence, $a = \lambda$.

Replace λ with a in the equation above, and we are left with

$$ax^2 + bx + c = a(x-\alpha_1)(x-\alpha_1)$$

Example. The discriminant of $3x^2 - 6x + 3x$ equals $(-6)^2 - 4(3)(3) = 36 - 36 = 0$, so there is exactly one root. We find the root using the quadratic formula:

$$\frac{-(-6)+\sqrt{0}}{2(3)} = \frac{6}{6} = 1$$

Therefore, $3x^2 - 6x + 3x$ is completely factored as $3(x-1)(x-1)$.

Summary. The following chart summarizes the discussion above.

roots of ax^2+bx+c	completely factored form of ax^2+bx+c
no roots	$a(x^2 + \frac{b}{a}x + \frac{c}{a})$
2 roots: α_1 and α_2	$a(x-\alpha_1)(x-\alpha_2)$
1 root: α_1	$a(x-\alpha_1)(x-\alpha_1)$

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Factors in $\mathbb{Z}$

Recall that the factors of an integer n are all of the integers k such that $n = mk$ for some third integer m.

Examples.

- $12 = 3 \cdot 4$, so 4 is a factor of 12.
- $-30 = -2 \cdot 15$, so 15 is a factor of -30.
- 1, -1, n and $-n$ are all factors of an integer n. That's because $n = n \cdot 1$ and $n = (-n)(-1)$.

Important special case. If $\alpha_1, \alpha_2, \dots \alpha_n \in \mathbb{Z}$, then each of these numbers are factors of the product $\alpha_1\alpha_2 \cdots \alpha_n$. For example, 2, 10, and 7 are each factors of $2 \cdot 10 \cdot 7 = 140$.

Check factors of degree 0 coefficient when searching for roots

If k, α_1, and α_2 are all integers, then the polynomial

$$q(x) = k(x - \alpha_1)(x - \alpha_2) = kx^2 - k(\alpha_1 + \alpha_2)x + k\alpha_1\alpha_2$$

has α_1 and α_2 as roots, and each of these roots are factors of the degree 0 coefficient of $q(x)$. (The degree 0 coefficient is $k\alpha_1\alpha_2$.)

More generally, if $k, \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{Z}$, then the degree 0 coefficient of the polynomial

$$g(x) = k(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$$

equals $k\alpha_1\alpha_2 \cdots \alpha_n$. That means that each of the roots of $g(x)$ – which are the α_i – are factors of the degree 0 coefficient of $g(x)$.

Now it's not true that every polynomial has integer roots, but many of the polynomials you will come across do, so the two paragraphs above offer a powerful hint as to what the roots of a polynomial might be.

When searching for roots of a polynomial
whose coefficients are all integers,
check the factors of the degree 0 coefficient.

Example. 3 and -7 are both roots of $2(x-3)(x+7)$.

Notice that $2(x-3)(x+7) = 2x^2 + 8x - 42$, and that 3 and -7 are both factors of -42.

Example. Suppose $p(x) = 3x^4 + 3x^3 - 3x^2 + 3x - 6$. This is a degree 4 polynomial, so it will have at most 4 roots. There isn't a really easy way to find the roots of a degree 4 polynomial, so to find the roots of $p(x)$, we have to start by guessing.

The degree 0 coefficient of $p(x)$ is -6, so a good place to check for roots is in the factors of -6.

The factors of -6 are 1, -1, 2, -2, 3, -3, 6, and -6, so we have eight quick candidates for what the roots of $p(x)$ might be. A quick check shows that of these eight candidates, exactly two are roots of $p(x)$ – namely 1 and -2. That is to say, $p(1) = 0$ and $p(-2) = 0$.

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Factoring cubics

It follows from the Fundamental Theorem of Algebra that a cubic polynomial is either the product of a constant and three linear polynomials, or else it is the product of a constant, one linear polynomial, and one quadratic polynomial that has no roots.

In either case, any cubic polynomial is guaranteed to have a linear factor, and thus is guaranteed to have a root. You're going to have to guess what that root is by looking at the factors of the degree 0 coefficient. (There is a "cubic formula" that like the quadratic formula will tell you the roots of a cubic, but the formula is difficult to remember, and you'd need to know about complex numbers to be able to use it.)

Once you've found a root, factor out the linear factor that the root gives you. You will now be able to write the cubic as a product of a monic linear polynomial and a quadratic polynomial. Completely factor the quadratic and then you will have completely factored the cubic.

Problem. Completely factor $2x^3 - 3x^2 + 4x - 3$.

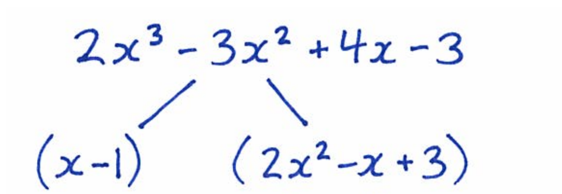
Solution. Start by guessing a root. The degree 0 coefficient is -3, and the factors of -3 are 1, -1, 3, and -3. Check these factors to see if any of them are roots.

After checking, you'll see that 1 is a root. That means that $x - 1$ is a factor of $2x^3 - 3x^2 + 4x - 3$. Therefore, we can divide $2x^3 - 3x^2 + 4x - 3$ by $x - 1$ to get another polynomial

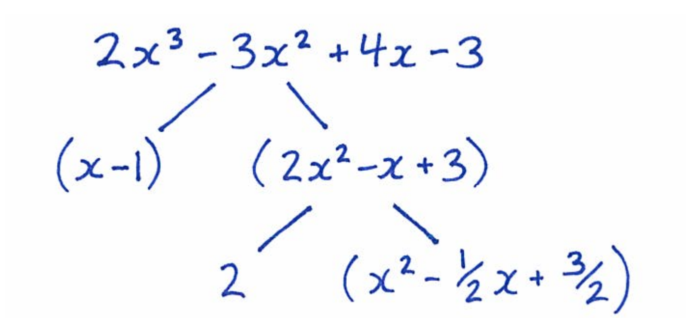
$$\frac{2x^3 - 3x^2 + 4x - 3}{x - 1} = 2x^2 - x + 3$$

Thus,

$$2x^3 - 3x^2 + 4x - 3 \;=\; (x-1)(2x^2 - x + 3)$$



The discriminant of $2x^2 - x + 3$ equals $(-1)^2 - 4(2)(3) = 1 - 24 = -23$, a negative number. Therefore, $2x^2 - x + 3$ has no roots, so to completely factor $2x^2 - x + 3$ we just have to factor out the leading coefficient as follows: $2x^2 - x + 3 = 2\big(x^2 - \frac{1}{2}x + \frac{3}{2}\big)$.



The final answer is

$$2(x-1)\Big(x^2 - \frac{1}{2}x + \frac{3}{2}\Big)$$

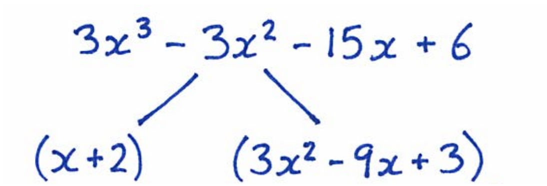
Problem. Completely factor $3x^3 - 3x^2 - 15x + 6$.

Solution. The factors of 6 are $\{1, -1, 2, -2, 3, -3, 6, -6\}$. Check to see that -2 is a root. Then divide by $x + 2$ to find that

$$\frac{3x^3 - 3x^2 - 15x + 6}{x+2} \; = \; 3x^2 - 9x + 3$$

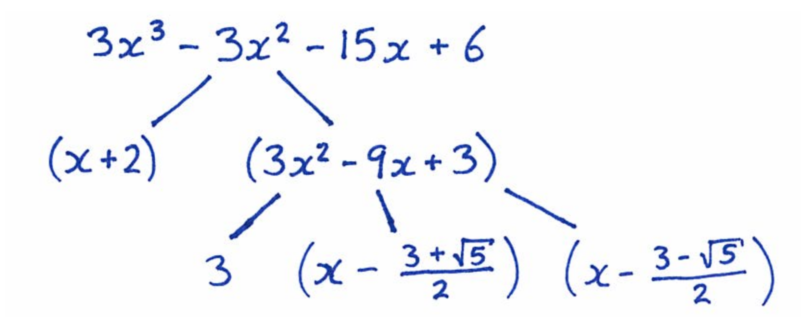
so

$$3x^3 - 3x^2 - 15x + 6 \; = \; (x+2)(3x^2 - 9x + 3)$$



The discriminant of $3x^2 - 9x + 3$ equals 45, and thus $3x^2 - 9x + 3$ has two roots and can be factored further.

The leading coefficient of $3x^2 - 9x + 3$ is 3, and we can use the quadratic formula to check that the roots of $3x^2 - 9x + 3$ are $\frac{3+\sqrt{5}}{2}$ and $\frac{3-\sqrt{5}}{2}$. From what we learned above about factoring quadratics, we know that $3x^2 - 9x + 3 = 3(x - \frac{3+\sqrt{5}}{2})(x - \frac{3-\sqrt{5}}{2})$.



To summarize,

$$\begin{aligned} 3x^3 - 3x^2 - 15x + 6 &= (x+2)(3x^2 - 9x + 3) \\ &= (x+2)3\Big(x - \frac{3+\sqrt{5}}{2}\Big)\Big(x - \frac{3-\sqrt{5}}{2}\Big) \\ &= 3(x+2)\Big(x - \frac{3+\sqrt{5}}{2}\Big)\Big(x - \frac{3-\sqrt{5}}{2}\Big) \end{aligned}$$

Factoring quartics

Degree 4 polynomials are tricky. As with cubic polynomials, you should begin by checking whether the factors of the degree 0 coefficient are roots. If one of them is a root, then you can use the same basic steps that we used with cubic polynomials to completely factor the polynomial.

The problem with degree 4 polynomials is that there's no reason that a degree 4 polynomial has to have any roots – take $(x^2+1)(x^2+1)$ for example.

Because a degree 4 polynomial might not have any roots, it might not have any linear factors, and it's very hard to guess which quadratic polynomials it might have as factors.

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Exercises

Completely factor the polynomials given in #1-8

1.) $10x + 20$

2.) $-2x + 5$

3.) $-2x^2 - 12x - 18$

4.) $10x^2 + 3$

5.) $3x^2 - 10x + 5$

6.) $3x^2 - 4x + 5$

7.) $-2x^2 + 6x - 3$

8.) $5x^2 + 3x - 2$

9.) Find a root of $x^3 - 5x^2 + 10x - 8$.

10.) Find a root of $15x^3 + 35x^2 + 30x + 10$.

11.) Find a root of $x^3 - 2x^2 - 2x - 3$.

Completely factor the polynomials in #12-16.

12.) $-x^3 - x^2 + x + 1$

13.) $5x^3 - 9x^2 + 8x - 20$

14.) $-2x^3 + 17x - 3$

15.) $4x^3 - 20x^2 + 25x - 3$

16.) $x^4 - 5x^2 + 4$

17.) How can the Fundamental Theorem of Algebra be used to show that any polynomial of odd degree has at least one root?