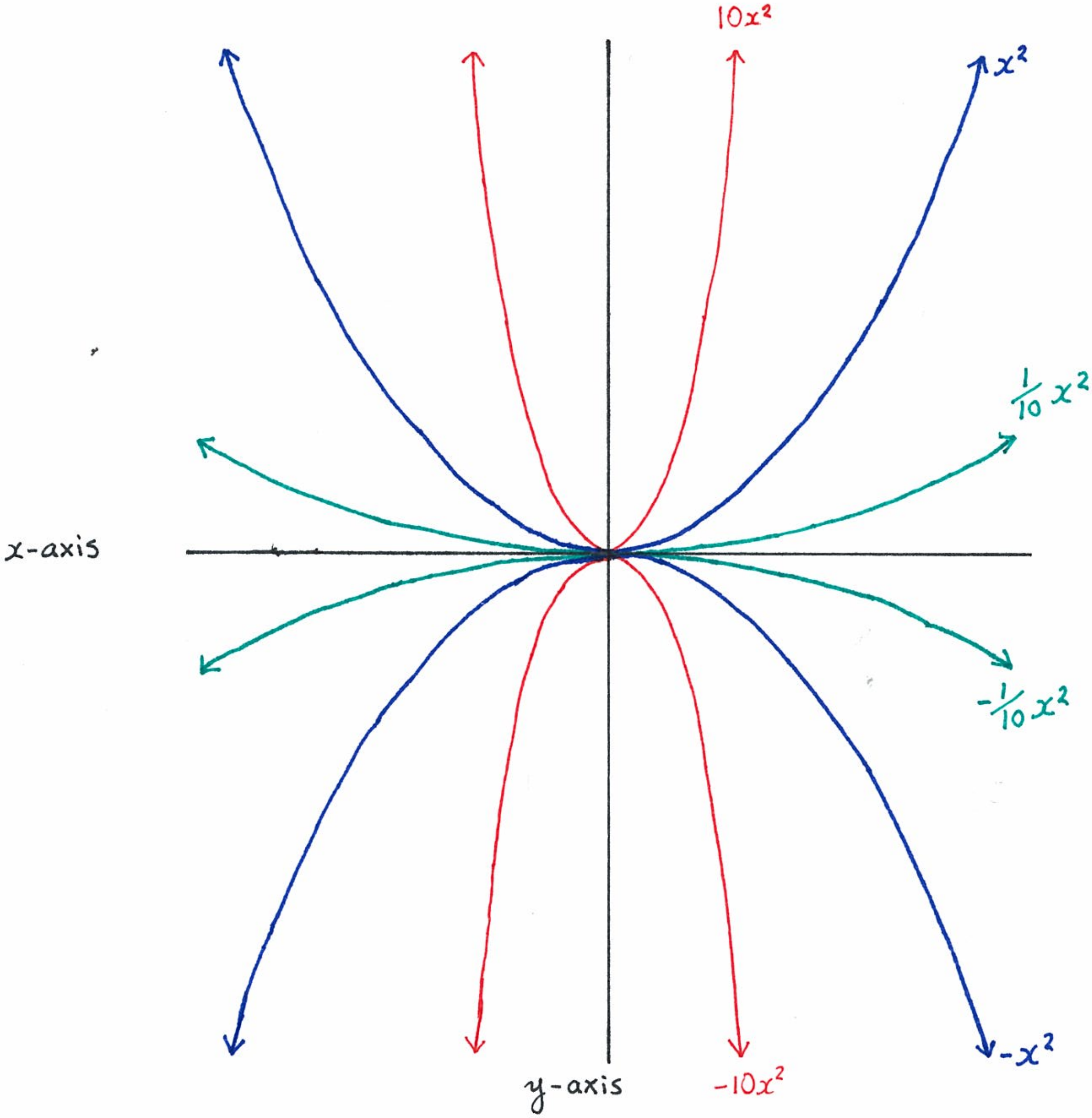
# Quadratic Polynomials

If $a > 0$ then the graph of $ax^2$ is obtained by starting with the graph of $x^2$, and then stretching or shrinking vertically by $a$.

If $a < 0$ then the graph of $ax^2$ is obtained by starting with the graph of $x^2$, then flipping it over the $x$-axis, and then stretching or shrinking vertically by the positive number $-a$.

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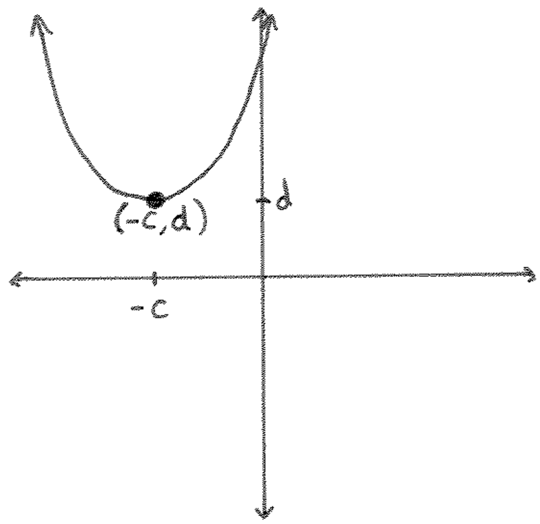
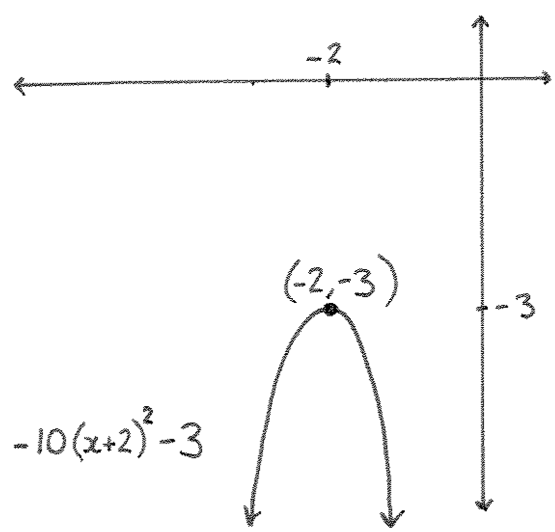
If $a, c, d \in \mathbb{R}$ and $a \neq 0$, then the graph of $a(x+c)^2 + d$ is obtained by shifting the graph of $ax^2$ horizontally by $c$, and vertically by $d$. (Remember that $d > 0$ means moving up, $d < 0$ means moving down, $c > 0$ means moving *left*, and $c < 0$ means moving *right*.)

If $a \neq 0$, the graph of a function $f(x) = a(x+c)^2 + d$ is called a *parabola*. The point $(-c, d) \in \mathbb{R}^2$ is called the *vertex* of the parabola.




**Example.** Below is the parabola that is the graph of $-10(x+2)^2 - 3$. Its vertex is $(-2, -3)$.

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A *quadratic polynomial* is a degree 2 polynomial. In other words, a quadratic polynomial is any polynomial of the form

$$p(x) = ax^2 + bx + c$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

## Completing the square

You should memorize this equation: (it's called *completing the square.*)

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$$ax^2 + bx + c \; = \; a\Big(x + \frac{b}{2a}\Big)^2 + c - \frac{b^2}{4a}$$

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Let's check that the equation is true:

$$\begin{aligned} a\Big(x + \frac{b}{2a}\Big)^2 + c - \frac{b^2}{4a} &= a\Big(x^2 + 2x\frac{b}{2a} + \Big[\frac{b}{2a}\Big]^2\Big) + c - \frac{b^2}{4a} \\ &= ax^2 + a2x\frac{b}{2a} + a\Big[\frac{b}{2a}\Big]^2 + c - \frac{b^2}{4a} \\ &= ax^2 + bx + a\Big[\frac{b^2}{4a^2}\Big] - \frac{b^2}{4a} + c \\ &= ax^2 + bx + \frac{b^2}{4a} - \frac{b^2}{4a} + c \\ &= ax^2 + bx + c \end{aligned}$$

## Graphing quadratics

We can use completing the square to graph quadratic polynomials. If $p(x) = ax^2 + bx + c$, then $p(x) = a\big(x + \frac{b}{2a}\big)^2 + c - \frac{b^2}{4a}$. Therefore, the graph of $p(x) = ax^2 + bx + c$ is obtained by shifting the graph of $ax^2$ horizontally by $\frac{b}{2a}$, and vertically by $c - \frac{b^2}{4a}$.

**Example.** To graph $-3x^2 + 5x - 2$, first complete the square to find that $-3x^2 + 5x - 2$ is the same polynomial as $-3(x - \frac{5}{6})^2 + \frac{1}{12}$. To graph this polynomial, we start with the parabola for $-3x^2$.

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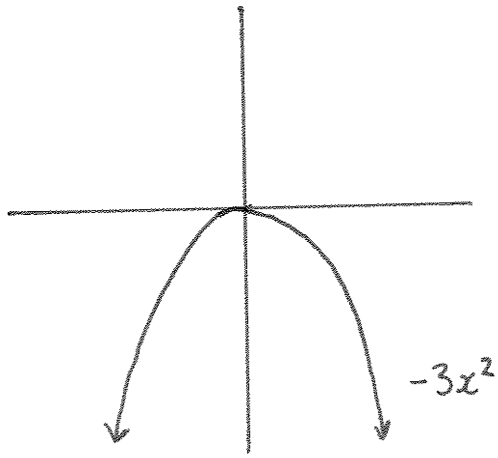
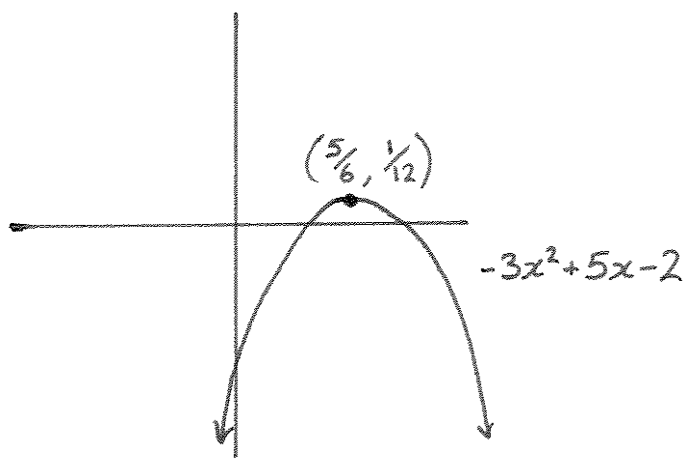



Shift the parabola for $3x^2$ right by $\frac{5}{6}$ and then up by $\frac{1}{12}$. The result is the graph for $-3x^2 + 5x - 2$. Notice that the graph looks like the graph of $-3x^2$, except that its vertex is the point $(\frac{5}{6}, \frac{1}{12})$.

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## Discriminant

The *discriminant* of $ax^2 + bx + c$ is defined to be the number $b^2 - 4ac$.

## How many roots?

If $p(x) = ax^2 + bx + c$, then the following chart shows how the discriminant of $p(x)$ determines how many roots $p(x)$ has:

| $b^2 - 4ac$ | number of roots |
|---|---|
| $> 0$ | 2 |
| $= 0$ | 1 |
| $< 0$ | 0 |

**Example.** Suppose $p(x) = -2x^2 + 3x - 1$. Because $3^2 - 4(-2)(-1) = 9 - 8 = 1$ is positive, $p(x) = -2x^2 + 3x - 1$ has two roots.

Why the discriminant of a quadratic polynomial tells us about the number of roots of the polynomial, and why the information from the above chart is true, will be explained in lectures.

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## Finding roots

If $ax^2 + bx + c$ has at least one root – which is the same as saying that $b^2 - 4ac \geq 0$ – then there is a formula that tells us what those roots are.

## Quadratic formula

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If $p(x) = ax^2 + bx + c$ with $a \neq 0$ and if $b^2 - 4ac \geq 0$, then the roots of $p(x)$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

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Notice in the quadratic formula, that we need $a \neq 0$ to make sure that we are not dividing by 0, and we need $b^2 - 4ac \geq 0$ to make sure that we aren't taking the square root of a negative number.

Also recall that if $ax^2 + bx + c$ has only one root, then $b^2 - 4ac = 0$. That means the two roots from the quadratic formula are really the same root.

It's a good exercise in algebra to check that the quadratic equation is true. To check that it's true, you need to check that

$$p\Big(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\Big) = 0$$

and that

$$p\Big(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\Big) = 0$$

**Example.** We checked above that $p(x) = -2x^2+3x-1$ has 2 roots, because its discriminant equalled 1. The quadratic formula tells us that those roots equal

$$\frac{-3 + \sqrt{1}}{2(-2)} = \frac{-3+1}{-4} = \frac{-2}{-4} = \frac{1}{2}$$

and

$$\frac{-3 - \sqrt{1}}{2(-2)} = \frac{-4}{-4} = 1$$

# Exercises

For each of the quadratic polynomials in problems #1-6:

- Complete the square.
- What's the vertex of the corresponding parabola?
- Is its parabola opening up, or opening down?
- What's its discriminant?
- How many roots does it have?
- What are its roots (if it has any)?
- Graph the polynomial, labeling its vertex and any $x$-intercepts.

1.) $-2x^2 - 2x + 12$

2.) $x^2 + 2x + 1$

3.) $3x^2 - 9x + 6$

4.) $-4x^2 + 16x - 19$

5.) $x^2 + 2x - 1$

6.) $3x^2 + 6x + 5$

7.) Suppose you shoot a feather straight up into the air, and that $t$ is the time measured in seconds that follow after you shoot the feather into the air. If the height of the feather at time $t$ is given by $-2t^2 + 20t$ feet, then what is the maximum height that the feather reaches? How many seconds does it take for the feather to reach its maximum height?

8.) Let's say you make cogs for a living. After accounting for the cost of building materials, you earn a profit of $x^2 - 10x + 45$ cents on the $x$-th cog that you make. Which cog do you earn the least amount of profit for making? How much profit do you earn for that cog?