Roots & Factors

Roots of a polynomial

A *root* of a polynomial $p(x)$ is a number $\alpha \in \mathbb{R}$ such that $p(\alpha) = 0$.

Examples.

- 3 is a root of the polynomial $p(x) = 2x - 6$ because
$$p(3) = 2(3) - 6 = 6 - 6 = 0$$
- 1 is a root of the polynomial $q(x) = 15x^2 - 7x - 8$ since
$$q(1) = 15(1)^2 - 7(1) - 8 = 15 - 7 - 8 = 0$$
- $(\sqrt[2]{2})^2 - 2 = 0$, so $\sqrt[2]{2}$ is a root of $x^2 - 2$.

Be aware: What we call a root is what others call a "real root", to emphasize that it is both a root and a real number. Since the only numbers we will consider in this course (and the only numbers considered in a basic calculus course) are real numbers, clarifying that a root is a "real root" won't be necessary.

Factors

A polynomial $q(x)$ is a *factor* of the polynomial $p(x)$ if there is a third polynomial $g(x)$ such that $p(x) = q(x)g(x)$.

Example. $3x^3 - x^2 + 12x - 4 = (3x - 1)(x^2 + 4)$, so $3x - 1$ is a factor of $3x^3 - x^2 + 12x - 4$. The polynomial $x^2 + 4$ is also a factor of $3x^3 - x^2 + 12x - 4$.

Factors and division

If you divide a polynomial $p(x)$ by another polynomial $q(x)$, and there is no remainder, then $q(x)$ is a factor of $p(x)$. That's because if there's no remainder, then $\frac{p(x)}{q(x)}$ is a polynomial, and $p(x) = q(x)\left(\frac{p(x)}{q(x)}\right)$. That's the definition of $q(x)$ being a factor of $p(x)$.

If $\frac{p(x)}{q(x)}$ has a remainder, then $q(x)$ is *not* a factor of $p(x)$.

Example. In the previous chapter we saw that
$$\frac{6x^2 + 5x + 1}{3x + 1} = 2x + 1$$

Multiplying the above equation by $3x + 1$ gives

$$6x^2 + 5x + 1 = (3x + 1)(2x + 1)$$

so $3x + 1$ is a factor of $6x^2 + 5x + 1$.

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Most important examples of roots

Notice that the number α is a root of the linear polynomial $x - \alpha$ since $\alpha - \alpha = 0$.

You have to be able to recognize these types of roots when you see them.

polynomial	root
$x - 2$	2
$x - 3$	3
$x - (-2)$	-2
$x + 2$	-2
$x + 15$	-15
$x - \alpha$	α

Linear factors give roots

Suppose there is some number α such that $x-\alpha$ is a factor of the polynomial $p(x)$. We'll see that α must be a root of $p(x)$.

That $x - \alpha$ is a factor of $p(x)$ means there is a polynomial $g(x)$ such that

$$p(x) = (x - \alpha)g(x)$$

Then

$$\begin{aligned} p(\alpha) &= (\alpha - \alpha)g(\alpha) \\ &= 0 \cdot g(\alpha) \\ &= 0 \end{aligned}$$

Notice that it didn't matter what polynomial $g(x)$ was, or what number $g(\alpha)$ was; α is a root of $p(x)$.

If $x - \alpha$ is a factor of $p(x)$,
then α is a root of $p(x)$.

Examples.

- 2 is a root of $p(x) = (x-2)(\pi^7 x^{15} - 27x^{11} + \frac{3}{4}x^5 - x^3)$ because

$$\begin{aligned} p(2) &= (2-2)(\pi^7 2^{15} - 27(2)^{11} + \frac{3}{4}2^5 - 2^3) \\ &= 0 \cdot (\pi^7 2^{15} - 27(2)^{11} + \frac{3}{4}2^5 - 2^3) \\ &= 0 \end{aligned}$$

- 4 is a root of $q(x) = (x-4)(x^{101} - x^{57} - 17x^3 + x)$
- -2, 1, and 5 are roots of the polynomial $3(x+2)(x-1)(x-5)$.

Roots give linear factors

Suppose the number α is a root of the polynomial $p(x)$. That means that $p(\alpha) = 0$. We'll see that $x - \alpha$ must be a factor of $p(x)$.

Let's start by dividing $p(x)$ by $(x-\alpha)$. Remember that when you divide a polynomial by a linear polynomial, the remainder is always a constant. So we'll get something that looks like

$$\frac{p(x)}{(x-\alpha)} = g(x) + \frac{c}{(x-\alpha)}$$

where $g(x)$ is a polynomial and $c \in \mathbb{R}$ is a constant.

Next we can multiply the previous equation by $(x-\alpha)$ to get

$$\begin{aligned} p(x) &= (x-\alpha)\Big(g(x) + \frac{c}{(x-\alpha)}\Big) \\ &= (x-\alpha)g(x) + (x-\alpha)\frac{c}{(x-\alpha)} \\ &= (x-\alpha)g(x) + c \end{aligned}$$

That means that

$$\begin{aligned} p(\alpha) &= (\alpha - \alpha)g(\alpha) + c \\ &= 0 \cdot g(\alpha) + c \\ &= 0 + c \\ &= c \end{aligned}$$

Now remember that $p(\alpha) = 0$. We haven't used that information in this problem yet, but we can now: because $p(\alpha) = 0$ and $p(\alpha) = c$, it must be that $c = 0$. Therefore,

$$p(x) = (x - \alpha)g(x) + c = (x - \alpha)g(x)$$

That means that $x - \alpha$ is a factor of $p(x)$, which is what we wanted to check.

If α is a root of $p(x)$,
then $x - \alpha$ is a factor of $p(x)$

Example. It's easy to see that 1 is a root of $p(x) = x^3 - 1$. Therefore, we know that $x - 1$ is a factor of $p(x)$. That means that $p(x) = (x - 1)g(x)$ for some polynomial $g(x)$.

To find $g(x)$, divide $p(x)$ by $x - 1$:

$$g(x) = \frac{p(x)}{x-1} = \frac{x^3 - 1}{x - 1} = x^2 + x + 1$$

Hence, $x^3 - 1 = (x - 1)(x^2 + x + 1)$.

We were able to find two factors of $x^3 - 1$ because we spotted that the number 1 was a root of $x^3 - 1$.

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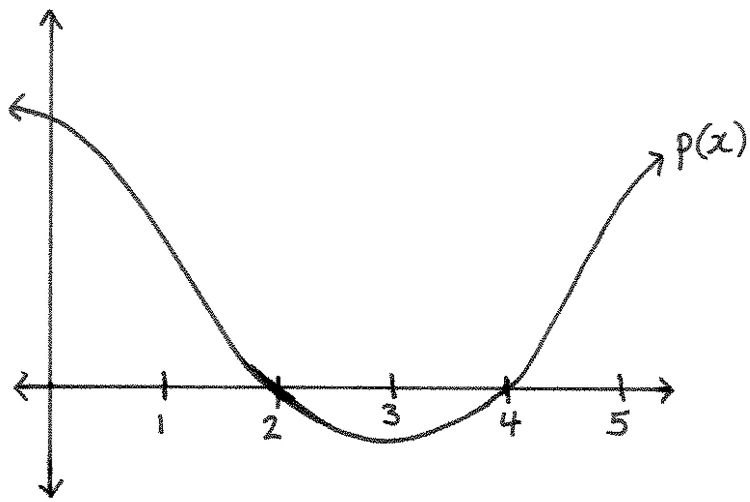
Roots and graphs

If you put a root into a polynomial, 0 comes out. That means that if α is a root of $p(x)$, then $(\alpha, 0) \in \mathbb{R}^2$ is a point in the graph of $p(x)$. These points are exactly the x-intercepts of the graph of $p(x)$.

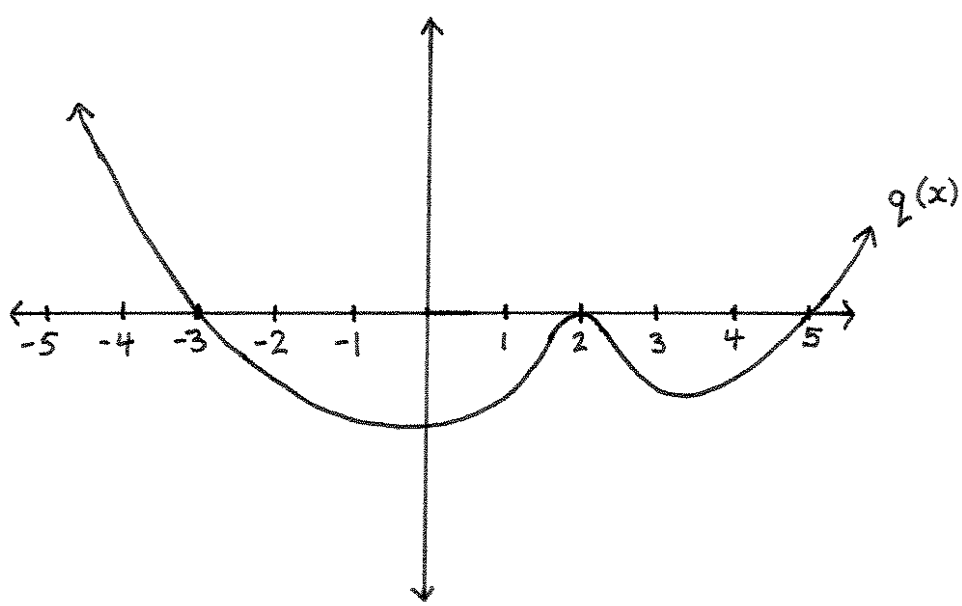
The roots of a polynomial are exactly the x-intercepts of its graph.

Examples.

• Below is the graph of a polynomial $p(x)$. The graph intersects the x-axis at 2 and 4, so 2 and 4 must be roots of $p(x)$. That means that $(x-2)$ and $(x-4)$ are factors of $p(x)$.



• Below is the graph of a polynomial $q(x)$. The graph intersects the x-axis at -3, 2, and 5, so -3, 2, and 5 are roots of $q(x)$, and $(x+3)$, $(x-2)$, and $(x-5)$ are factors of $q(x)$.



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Degree of a product is the sum of degrees of the factors

Let's take a look at some products of polynomials that we saw before in the chapter on "Basics of Polynomials":

The leading term of $(2x^2 - 5x)(-7x + 4)$ is $-14x^3$. This is an example of a degree 2 and a degree 1 polynomial whose product equals 3. Notice that $2 + 1 = 3$

The product $5(x-2)(x+3)(x^2+3x-7)$ is a degree 4 polynomial because its leading term is $5x^4$. The degrees of 5, $(x-2)$, $(x+3)$, and (x^2+3x-7) are 0, 1, 1, and 2, respectively. Notice that $0+1+1+2=4$.

The degrees of $(2x^3-7)$, (x^5-3x+5), $(x-1)$, and $(5x^7+6x-9)$ are 3, 5, 1, and 7, respectively. The degree of their product,

$$(2x^3-7)(x^5-3x+5)(x-1)(5x^7+6x-9),$$

equals 16 since its leading term is $10x^{16}$. Once again, we have that the sum of the degrees of the factors equals the degree of the product: $3+5+1+7=16$.

These three examples suggest a general pattern that always holds for factored polynomials (as long as the factored polynomial does not equal 0):

If a polynomial $p(x)$ is factored into a product of polynomials, then the degree of $p(x)$ equals the sum of the degrees of its factors.

Examples.

- The degree of $(4x^3+27x-3)(3x^6-27x^3+15)$ equals $3+6=9$.
- The degree of $-7(x+4)(x-1)(x-3)(x-3)(x^2+1)$ equals $0+1+1+1+1+2=6$.

Degree of a polynomial bounds the number of roots

Suppose $p(x)$ is a polynomial that has n roots, and that $p(x)$ is not the constant polynomial $p(x)=0$. Let's name the roots of $p(x)$ as $\alpha_1, \alpha_2, ..., \alpha_n$.

Any root of $p(x)$ gives a linear factor of $p(x)$, so

$$p(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)q(x)$$

for some polynomial $q(x)$.

Because the degree of a product is the sum of the degrees, the degree of $p(x)$ is at least n.

The degree of $p(x)$ (if $p(x) \neq 0$) is greater than or equal to the number of roots that $p(x)$ has.

Examples.

- $5x^4 - 3x^3 + 2x - 17$ has at most 4 roots.
- $4x^{723} - 15x^{52} + 37x^{14} - 7$ has at most 723 roots.
- Aside from the constant polynomial $p(x) = 0$, if a function has a graph that has infinitely many x-intercepts, then the function cannot be a polynomial.

  If it were a polynomial, its number of roots (or alternatively, its number of x-intercepts) would be bounded by the degree of the polynomial, and thus there would only be finitely many x-intercepts.

  To illustrate, if you are familiar with the graphs of the functions $\sin(x)$ and $\cos(x)$, then you'll recall that they each have infinitely many x-intercepts. Thus, they cannot be polynomials. (If you are unfamiliar with $\sin(x)$ and $\cos(x)$, then you can ignore this paragraph.)

Exercises

1.) Name two roots of the polynomial $(x-1)(x-2)$.

2.) Name two roots of the polynomial $-(x+7)(x-3)(x^4+x^3+2x^2+x+1)$.

3.) Name four roots of the polynomial $-\frac{2}{5}(x+\frac{7}{3})(x+\frac{1}{2})(x-\frac{4}{3})(x-\frac{9}{2})(x^2+1)$.

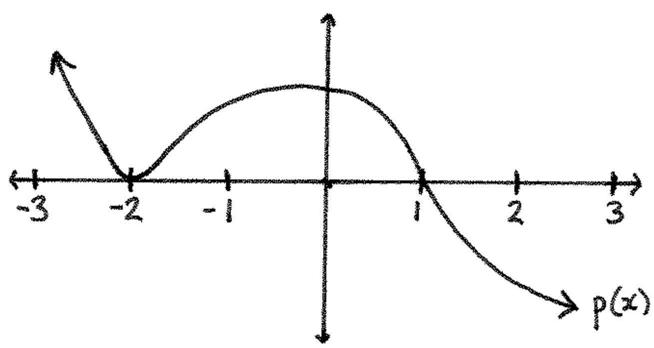
It will help with #4-6 to know that each of the polynomials from those problems has a root that equals either -1, 0, or 1.

4.) Write x^3+4x-5 as a product of a linear and a quadratic polynomial.

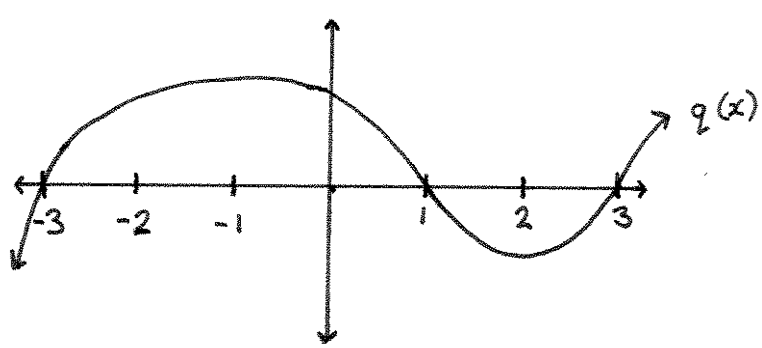
5.) Write x^3+x as a product of a linear and a quadratic polynomial. (Hint: you could use the distributive law here.)

6.) Write $x^5+3x^4+x^3-x^2-x-1$ as a product of a linear and a quartic polynomial.

7.) The graph of a polynomial $p(x)$ is drawn below. Identify as many roots and factors of $p(x)$ as you can.



8.) The graph of a polynomial $q(x)$ is drawn below. Identify as many roots and factors of $q(x)$ as you can.



For #9-13, determine the degree of the given polynomial.

9.) $(x+3)(x-2)$

10.) $(3x+5)(4x^2+2x-3)$

11.) $-17(3x^2+20x-4)$

12.) $4(x-1)(x-1)(x-1)(x-2)(x^2+7)(x^2+3x-4)$

13.) $5(x-3)(x^2+1)$

14.) (True/False) $7x^5+13x^4-3x^3-7x^2+2x-1$ has 8 roots.