Inverse Functions

One-to-one

Suppose $f : A \to B$ is a function. We call f *one-to-one* if every distinct pair of objects in A is assigned to a distinct pair of objects in B. In other words, each object of the target has at most one object from the domain assigned to it.

There is a way of phrasing the previous definition in a more mathematical language: f is one-to-one if whenever we have two objects $a, c \in A$ with $a \neq c$, we are guaranteed that $f(a) \neq f(c)$.

Example. $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$ is not one-to-one because $3 \neq -3$ and yet $f(3) = f(-3)$ since $f(3)$ and $f(-3)$ both equal 9.
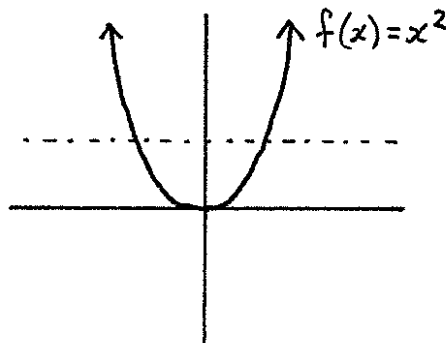
Horizontal line test

If a horizontal line intersects the graph of $f(x)$ in more than one point, then $f(x)$ is not one-to-one.

The reason $f(x)$ would not be one-to-one is that the graph would contain two points that have the same second coordinate – for example, $(2, 3)$ and $(4, 3)$. That would mean that $f(2)$ and $f(4)$ both equal 3, and one-to-one functions can't assign two different objects in the domain to the same object of the target.

If *every* horizontal line in $\mathbb{R}^2$ intersects the graph of a function *at most once*, then the function is one-to-one.

Examples. Below is the graph of $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$. There is a horizontal line that intersects this graph in more than one point, so f is not one-to-one.

Below is the graph of $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = x^3$. Any horizontal line that could be drawn would intersect the graph of g in at most one point, so g is one-to-one.
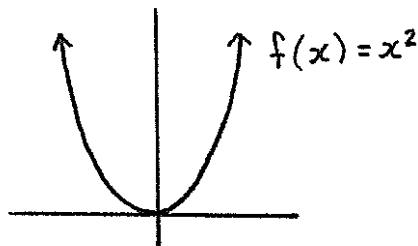
$$g(x) = x^3$$
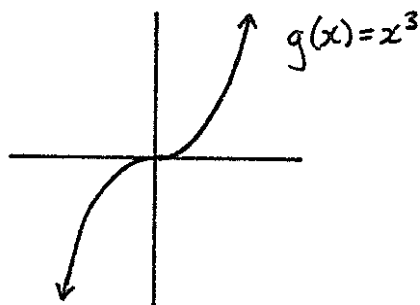
Onto

Suppose $f : A \to B$ is a function. We call f *onto* if the range of f equals B.

In other words, f is onto if every object in the target has at least one object from the domain assigned to it by f.

Examples. Below is the graph of $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$. Using techniques learned in the chapter "Intro to Graphs", we can see that the range of f is $[0, \infty)$. The target of f is $\mathbb{R}$, and $[0, \infty) \neq \mathbb{R}$ so f is not onto.

$$f(x) = x^2$$

Below is the graph of $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = x^3$. The function g has the set $\mathbb{R}$ for its range. This equals the target of g, so g is onto.

$$g(x) = x^3$$

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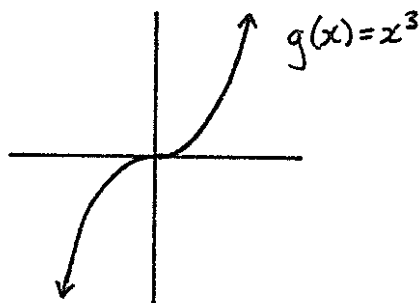
What an inverse function is

Suppose $f : A \to B$ is a function. A function $g : B \to A$ is called the *inverse function* of f if $f \circ g = id$ and $g \circ f = id$.

If g is the inverse function of f, then we often rename g as f^{-1}.

Examples.

- Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x + 3$, and let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x) = x - 3$. Then

$$f \circ g(x) = f(g(x)) = f(x - 3) = (x - 3) + 3 = x$$

Because $f \circ g(x) = x$ and $id(x) = x$, these are the same function. In symbols, $f \circ g = id$.

Similarly

$$g \circ f(x) = g(f(x)) = g(x + 3) = (x + 3) - 3 = x$$

so $g \circ f = id$. Therefore, g is the inverse function of f, so we can rename g as f^{-1}, which means that $f^{-1}(x) = x - 3$.

- Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 2x + 2$, and let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x) = \frac{1}{2}x - 1$. Then

$$f \circ g(x) = f(g(x)) = f\left(\frac{1}{2}x - 1\right) = 2\left(\frac{1}{2}x - 1\right) + 2 = x$$

Similarly

$$g \circ f(x) = g(f(x)) = g(2x + 2) = \frac{1}{2}\left(2x + 2\right) - 1 = x$$

Therefore, g is the inverse function of f, which means that $f^{-1}(x) = \frac{1}{2}x - 1$.

The Inverse of an inverse is the original

If f^{-1} is the inverse of f, then $f^{-1} \circ f = id$ and $f \circ f^{-1} = id$. We can see from the definition of inverse functions above, that f is the inverse of f^{-1}. That is $(f^{-1})^{-1} = f$.

Inverse functions "reverse the assignment"

The definition of an inverse function is given above, but the essence of an inverse function is that it reverses the assignment dictated by the original function. If f assigns a to b, then f^{-1} will assign b to a. Here's why:

If $f(a) = b$, then we can apply f^{-1} to both sides of the equation to obtain the new equation $f^{-1}(f(a)) = f^{-1}(b)$. The left side of the previous equation involves function composition, $f^{-1}(f(a)) = f^{-1} \circ f(a)$, and $f^{-1} \circ f = id$, so we are left with $f^{-1}(b) = id(a) = a$.

The above paragraph can be summarized as "If $f(a) = b$, then $f^{-1}(b) = a$."

Examples.

- If $f(3) = 4$, then $3 = f^{-1}(4)$.

- If $f(-2) = 16$, then $-2 = f^{-1}(16)$.

- If $f(x + 7) = -1$, then $x + 7 = f^{-1}(-1)$.

- If $f^{-1}(0) = -4$, then $0 = f(-4)$.

- If $f^{-1}(x^2 - 3x + 5) = 3$, then $x^2 - 3x + 5 = f(3)$.

In the 5 examples above, we "erased" a function from the left side of the equation by applying its inverse function to the right side of the equation.

When a function has an inverse

A function has an inverse exactly when it is *both* one-to-one and onto. This will be explained in more detail during lecture.

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Using inverse functions

Inverse functions are useful in that they allow you to "undo" a function. Below are some rather abstract (though important) examples. As the semester continues, we'll see some more concrete examples.

Examples.

● Suppose there is an object in the domain of a function f, and that this object is named a. Suppose that you know $f(a) = 15$.

If f has an inverse function, f^{-1}, and you happen to know that $f^{-1}(15) = 3$, then you can solve for a as follows: $f(a) = 15$ implies that $a = f^{-1}(15)$. Thus, $a = 3$.

● If b is an object of the domain of g, g has an inverse, $g(b) = 6$, and $g^{-1}(6) = -2$, then

$$b = g^{-1}(6) = -2$$

● Suppose $f(x + 3) = 2$. If f has an inverse, and $f^{-1}(2) = 7$, then

$$x + 3 = f^{-1}(2) = 7$$

so

$$x = 7 - 3 = 4$$

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# The Graph of an inverse

If f is an *invertible* function (that means if f has an inverse function), and if you know what the graph of f looks like, then you can draw the graph of f^{-1}.

If (a, b) is a point in the graph of $f(x)$, then $f(a) = b$. Hence, $f^{-1}(b) = a$. That means f^{-1} assigns b to a, so (b, a) is a point in the graph of $f^{-1}(x)$.

Geometrically, if you switch all the first and second coordinates of points in $\mathbb{R}^2$, the result is to flip $\mathbb{R}^2$ over the "$x = y$ line".

New function	How points in graph of $f(x)$ become points of new graph	visual effect
$f^{-1}(x)$	$(a, b) \mapsto (b, a)$	flip over the "$x = y$ line"

Example.

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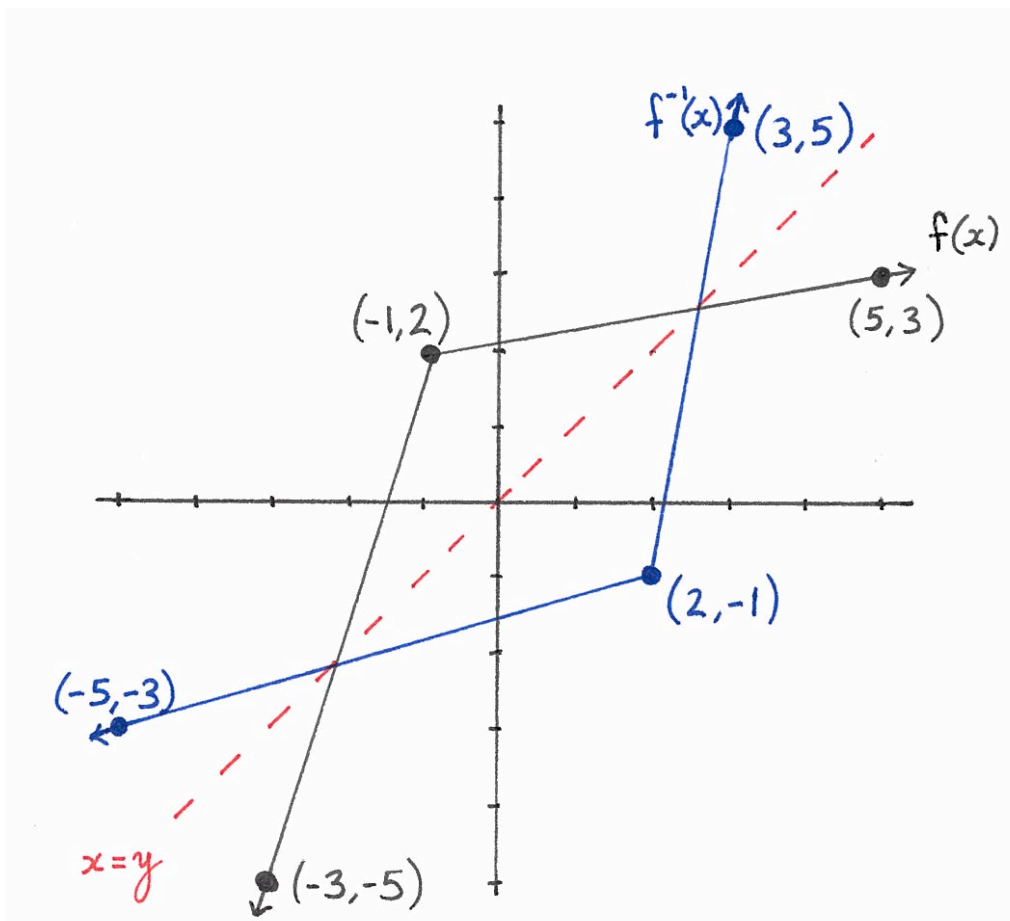
How to find an inverse

If you know that f is an invertible function, and you have an equation for $f(x)$, then you can find the equation for f^{-1} in three steps.

Step 1 is to replace $f(x)$ with the letter y.

Step 2 is to use algebra to solve for x.

Step 3 is to replace x with $f^{-1}(y)$.

After using these three steps, you'll have an equation for the function $f^{-1}(y)$.

Examples.

- Find the inverse of $f(x) = x + 5$.

$$\text{Step 1.} \qquad y = x + 5$$

$$\text{Step 2.} \qquad x = y - 5$$

$$\text{Step 3.} \qquad f^{-1}(y) = y - 5$$

- Find the inverse of $g(x) = \frac{2x}{x-1}$.

$$\text{Step 1.} \qquad y = \frac{2x}{x-1}$$

$$\text{Step 2.} \qquad x = \frac{y}{y-2}$$

$$\text{Step 3.} \qquad g^{-1}(y) = \frac{y}{y-2}$$

Make sure that you are comfortable with the algebra required to carry out step 2 in the above problem. You will be expected to perform similar algebra on future exams.

You should also be able to check that $g \circ g^{-1} = id$ and that $g^{-1} \circ g = id$.

Exercises

In #1-6, g is an invertible function.

1.) If $g(2) = 3$, what is $g^{-1}(3)$?

2.) If $g(7) = -2$, what is $g^{-1}(-2)$?

3.) If $g(-10) = 5$, what is $g^{-1}(5)$?

4.) If $g^{-1}(6) = 8$, what is $g(8)$?

5.) If $g^{-1}(0) = 9$, what is $g(9)$?

6.) If $g^{-1}(4) = 13$, what is $g(13)$?

For #7-12, solve for x. Use that f is an invertible function and that

$$f^{-1}(1) = -2$$
$$f^{-1}(2) = 3$$
$$f^{-1}(3) = 2$$
$$f^{-1}(4) = 5$$
$$f^{-1}(5) = -7$$
$$f^{-1}(6) = 8$$
$$f^{-1}(7) = -3$$
$$f^{-1}(8) = 1$$
$$f^{-1}(9) = 4$$

7.) $f(x + 2) = 5$

8.) $f(3x - 4) = 3$

9.) $f(-5x) = 1$

10.) $f(-2 - x) = 2$

11.) $f(\frac{1}{x}) = 8$

12.) $f(\frac{5}{x-1}) = 3$

Each of the functions given in #13-18 is invertible. Find the equations for their inverse functions.

13.) $f(x) = 3x + 2$
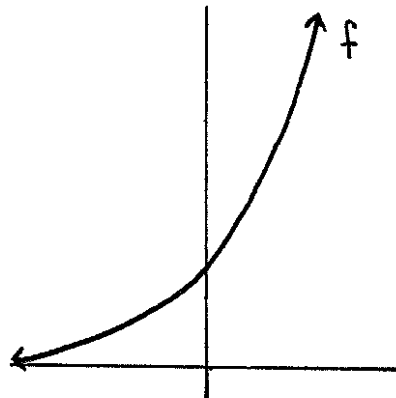
14.) $g(x) = -x + 5$

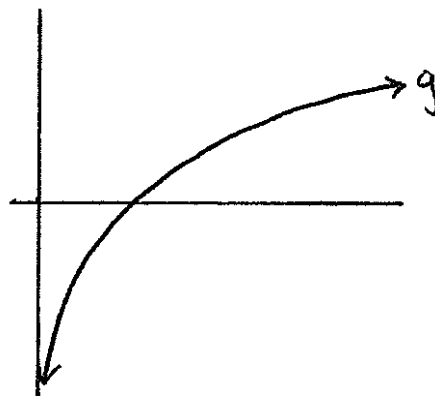15.) $h(x) = \frac{1}{x}$

16.) $f(x) = \frac{x}{x-1}$
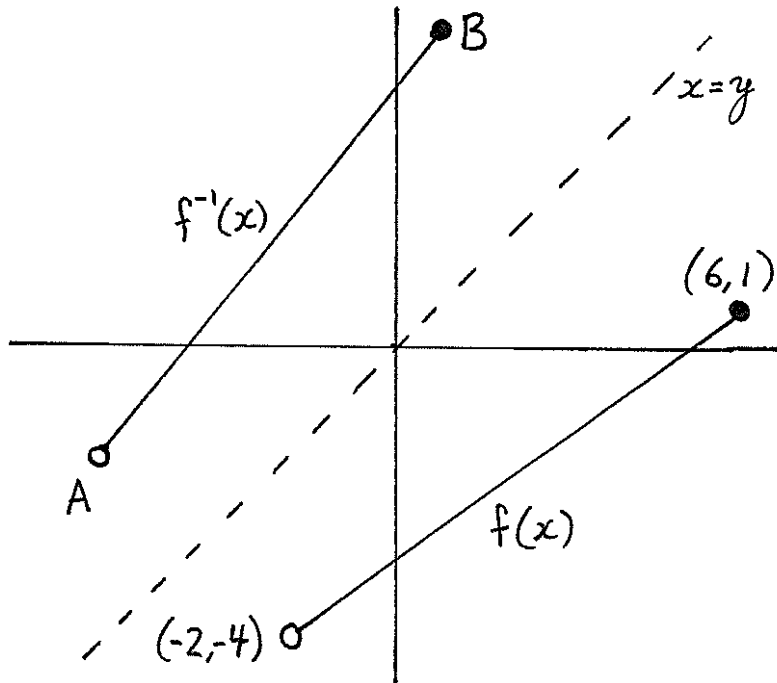
17.) $g(x) = \frac{2x+3}{x}$

18.) $h(x) = \frac{x}{4-x}$

19.) Below is the graph of $f : \mathbb{R} \to (0, \infty)$. Does f have an inverse?

20.) Below is the graph of $g : (0, \infty) \to \mathbb{R}$. Does g have an inverse?

21.) Below are the graphs of $f(x)$ and $f^{-1}(x)$. What are the coordinates of the points A and B on the graph of $f^{-1}(x)$?

22.) Below are the graphs of $g(x)$ and $g^{-1}(x)$. What are the coordinates of the points C and D on the graph of $g(x)$?