Homework 5 Solutions

N-th Roots : 1-4, 7-10, 14, 15, 18-26

1) $(x+7)^3 = 8$

$$x+7 = \sqrt[3]{8} = 2$$

$$\boxed{x = -5}$$

2) $\sqrt[2]{x+2} = 4$

$$x+2 = (4)^2 = 16$$

$$\boxed{x = 14}$$

3) $4(2x+7)^5 = 12$

$$(2x+7)^5 = 3$$

$$2x+7 = \sqrt[5]{3}$$

$$2x = \sqrt[5]{3} - 7$$

$$\boxed{x = \frac{\sqrt[5]{3} - 7}{2}}$$

4) $3\sqrt[4]{4-x} = 9$

$$\sqrt[4]{4-x} = 3$$

$$4-x = (3)^4 = 81$$

$$-x = 77$$

$$\boxed{x = -77}$$

7) $2x - 13 < 4$

$2x < 17$

$\boxed{x < \frac{17}{2}}$

8) $-3x < 16 + x$

$-4x < 16$ note: sign switches

$\boxed{x > -4}$

9) $\frac{4}{x} > \frac{1}{9}$

case 1: $x > 0$ then $\frac{4}{x} > \frac{1}{9}$

$4 > \frac{x}{9}$ $\Rightarrow$ $0 < x < 36$

$36 > x$

case 2: $x < 0$ then $\frac{4}{x}$ is negative and can't be greater then $\frac{1}{9}$

$\Rightarrow$ $\boxed{0 < x < 36}$

10) $\sqrt[5]{2x - 6} > 2$

$2x - 6 > (2)^5 = 32$

$2x > 38$

$\boxed{x > 19}$

14) $f(x) = \sqrt[15]{3x^2 - 14x + 9}$

no division by 0

15 is odd $\Rightarrow$ no even roots of negatives

$\Rightarrow$ Domain $= \mathbb{R}$

15) $g(x) = \sqrt[2]{17 - 2x}$

no division by 0

2 is an even root, so we need $17 - 2x \geq 0$

$-2x \geq -17$

$x \leq \frac{17}{2}$

$\Rightarrow$ Domain $= \left(-\infty, \frac{17}{2}\right]$

18) $\sqrt{27} = \sqrt{3^3} = \sqrt{3^2} \cdot \sqrt{3} = 3\sqrt{3}$

19) $\sqrt{24} = \sqrt{2^3 \cdot 3} = \sqrt{2^2} \cdot \sqrt{6} = 2\sqrt{6}$

20) $\sqrt{100} = \sqrt{10^2} = 10$

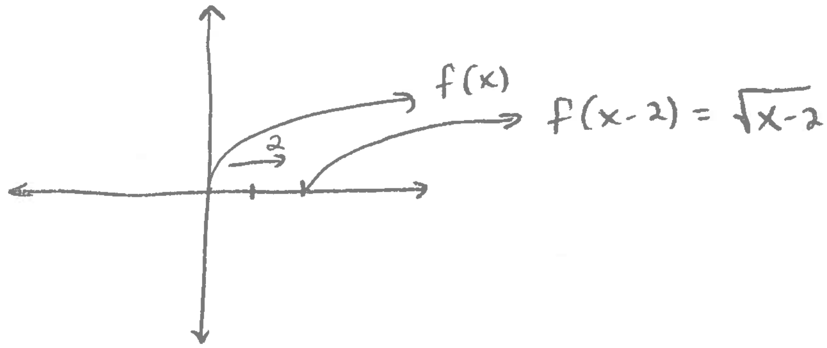
21) $\sqrt{52} = \sqrt{4 \cdot 13} = \sqrt{2^2} \cdot \sqrt{13} = 2\sqrt{13}$

22) $\sqrt{150} = \sqrt{3 \cdot 50} = \sqrt{3 \cdot 5 \cdot 2 \cdot 5} = \sqrt{5^2}\sqrt{6} = 5\sqrt{6}$

23) $\sqrt{48} = \sqrt{4 \cdot 12} = \sqrt{2^2}\sqrt{12} = 2\sqrt{12} = 2\sqrt{4 \cdot 3} = 2\sqrt{2^2}\sqrt{3} = 4\sqrt{3}$

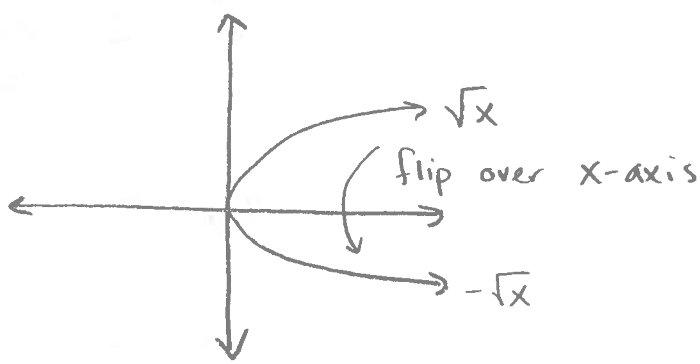
24) Graph $\sqrt{x-2}$ consider $\sqrt{x} = f(x)$

then $\sqrt{x-2} = f(x-2)$ is a graph transformation of $f(x)$

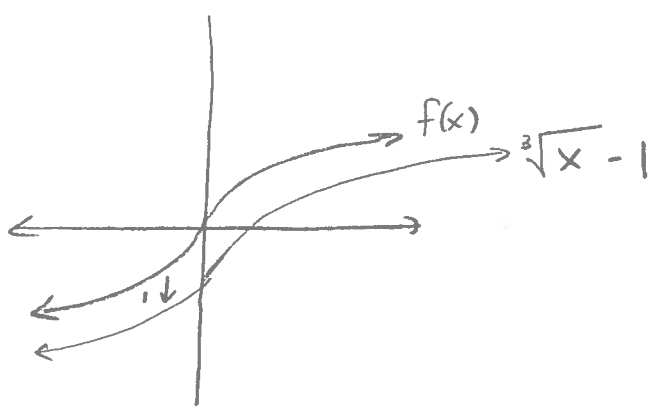


25) Graph $-\sqrt{x}$ note if $f(x) = \sqrt{x}$

$-\sqrt{x} = -f(x)$



26) $\sqrt[3]{x} - 1$ Let $f(x) = \sqrt[3]{x}$, then $\sqrt[3]{x} - 1 = f(x) - 1$



Factoring Polynomials: 1-4, 9-11, 12, 14, 16, 17

1) $10x + 20$ — not monic, factor out 10

$$\boxed{10(x+2)}$$

2) $-2x + 5$ — not monic, factor out -2

$$\boxed{-2\left(x - \frac{5}{2}\right)}$$

3) $-2x^2 - 12x - 18$ — not monic, factor out -2

$$-2(x^2 + 6x + 9)$$

Check discriminant on $x^2 + 6x + 9$

$$b^2 - 4ac = 36 - 4(1)(9) = 0$$

$\Rightarrow$ 1 double root

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6}{2(1)} = -3$$

$\Rightarrow$ $(x+3)$ is a double factor

$$\boxed{-2(x+3)(x+3)}$$

4) $10x^2 + 3$ — not monic, factor out 10

$$\boxed{10\left(x^2 + \frac{3}{10}\right)}$$

Check discriminant on $x^2 + \frac{3}{10}$

$$b^2 - 4ac = 0 - 4(1)\left(\frac{3}{10}\right) = -\frac{12}{10} < 0$$

$\Rightarrow$ no roots

9) Find a root of $x^3 - 5x^2 + 10x - 8$

$\boxed{x=2}$ is a root,

$$(2)^3 - 5(2)^2 + 10(2) - 8$$

$$= 8 - 20 + 20 - 8 = 0 \checkmark$$

10) $15x^3 + 35x^2 + 30x + 10$

$\boxed{x=-1}$ is a root

$$15(-1)^3 + 35(-1)^2 + 30(-1) + 10$$

$$= -15 + 35 - 30 + 10 = 45 - 45 = 0 \checkmark$$

11) $x^3 - 2x^2 - 2x - 3$

$\boxed{x=3}$ is a root

$$(3)^3 - 2(3)^2 - 2(3) - 3$$

$$27 - 18 - 6 - 3 = 27 - 27 = 0 \checkmark$$

12) Completely factor

$-x^3 - x^2 + x + 1$

guess a root, $x=1$ works

$\Rightarrow$ $x-1$ is a factor

$$\begin{array}{r l}
 & -x^2 - 2x - 1 \\
x-1 & \overline{)\,-x^3 - x^2 + x + 1} \\
 & -\underline{(-x^3 + x^2)} \\
 & -2x^2 + x \\
 & -\underline{(-2x^2 + 2x)} \\
 & -x + 1
\end{array}$$

$$\Rightarrow -x^3 - x^2 + x + 1 = (x-1)(-x^2 - 2x - 1)$$

$-x^2 - 2x - 1$ is not monic, factor out -1

$\Rightarrow \; -(x-1)(x^2+2x+1)$

check discriminant of x^2+2x+1

$b^2 - 4ac = 4 - 4(1)(1) = 0$

$\Rightarrow$ 1 double root

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2}{2(1)} = -1$$

$\Rightarrow \; (x+1)$ is a double factor

$$\boxed{-(x-1)(x+1)(x+1)}$$

14) Completely factor

$-2x^3 + 17x - 3$

guess a root: $x = -3$ work

$-2(-3)^3 + 17(-3) - 3$

$= -2(-27) - 51 - 3 = 54 - 54 = 0$

then $(x+3)$ is a factor

$$\begin{array}{r l}
 & -2x^2 + 6x - 1 \\
x+3 \,\big) & -2x^3 + 17x - 3 \\
 & -(-2x^3 - 6x^2) \\
 & 6x^2 + 17x \\
 & -(6x^2 + 18x) \\
 & -x - 3 \\
 & -(-x - 3) \\
 & 0
\end{array}$$

thus $\quad -2x^3 + 17x - 3 = (x+3)(-2x^2 + 6x - 1)$

$-2x^2 + 6x - 1$ is not monic, factor out -2

$-2(x+3)(x^2 - 3x + \frac{1}{2})$

check discriminant:

$b^2 - 4ac = (-3)^2 - 4(1)(\frac{1}{2})$

$= 9 - 2 = 7 > 0$

$\Rightarrow$ 2 roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm \sqrt{7}}{2}$$

roots: $\frac{3+\sqrt{7}}{2}$, $\frac{3-\sqrt{7}}{2}$

$\Rightarrow$ factors: $\left(x - \frac{3+\sqrt{7}}{2}\right)$, $\left(x - \frac{3-\sqrt{7}}{2}\right)$

$$\Rightarrow \boxed{-2(x+3)\left(x - \frac{3+\sqrt{7}}{2}\right)\left(x - \frac{3-\sqrt{7}}{2}\right)}$$

16) $x^4 - 5x^2 + 4$ let $y = x^2$

$y^2 - 5y + 4$

$(y-4)(x-1)$ substitute back

$(x^2-4)(x^2-1)$

$= \boxed{(x+2)(x-2)(x+1)(x-1)}$

$\Rightarrow x^2 - 4 = 0$ $x = \pm 2$ roots

$(x+2)(x-2)$ factors

$\Rightarrow x^2 - 1 = 0$ $x = \pm 1$ roots

$(x+1)(x-1)$ factors

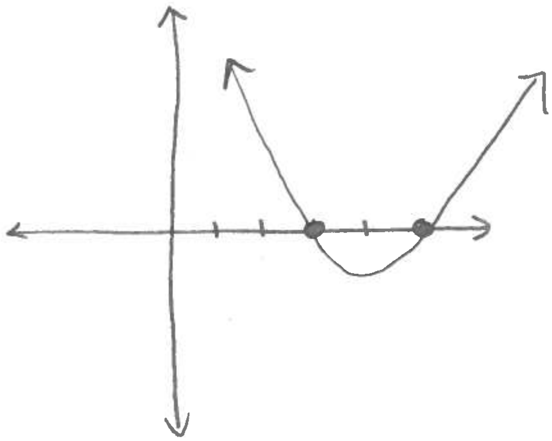
17) The fundamental theorem of algebra states that every polynomial can be expressed as a product of a real number (degree 0), a collection of monic linear factors (degree 1), and monic quadratic factors with no roots (degree 2). To get an odd degree product we therefore need at least one linear factor. This gives at least one root.

Graphing Polynomials: 1-4, 7, 8, 11

1) $4(x-3)(x-5)$

Leading Order Term = $4x^2$

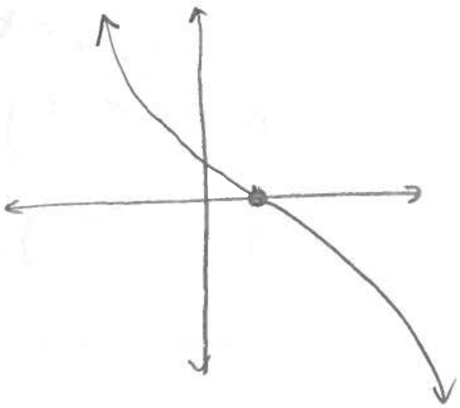
roots = 3, 5



2) $-(x-1)(x^2+x+5)$

Leading order term = $-x^3$

roots = 1



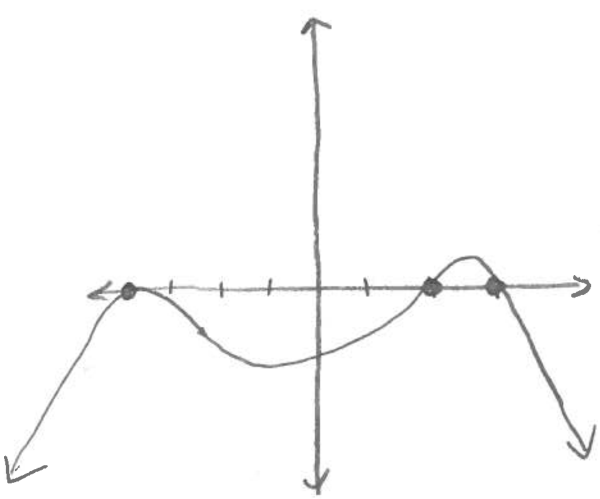
3) $-6(x+4)(x+4)(x-2)(x-3)(x^2+1)(x^2+3)$

Leading Order Term = $-6x^8$

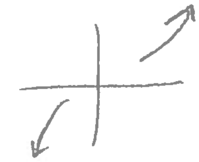
roots = -4, 2, 3

$\uparrow$

double root



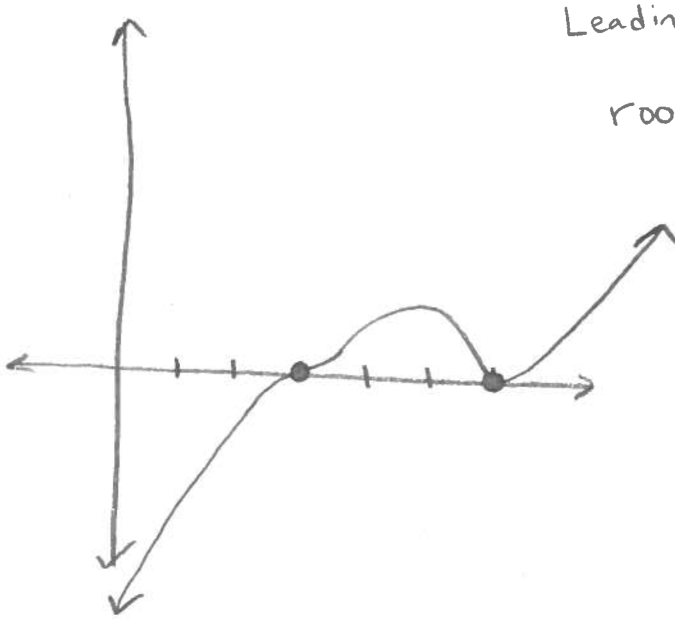
4) $2(x-3)(x-3)(x-3)(x-6)(x-6)(x^2+2x+7)$



Leading Order Term $= 2x^7$

roots $= 3, 6$

triple root — double root



7) $2x^2 + 2x - 24$

not monic, factor out 2

$2(x^2 + x - 12)$

$2(x+4)(x-3)$

discriminant $= b^2 - 4ac$

$= 1^2 - 4(1)(12) = 49$

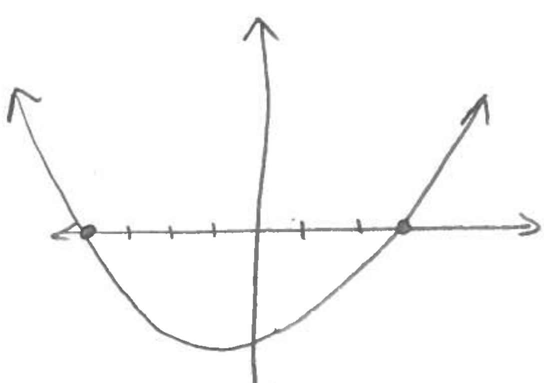
$x = \frac{-1 \pm \sqrt{49}}{2(1)} = \frac{-1 \pm 7}{2}$

$x = -4, 3$ roots

$\Rightarrow$ factors $= (x+4), (x-3)$

Leading Order term $= 2x^2$

roots $= -4, 3$



8) $7x^2 - 3x + 4$

check discriminant

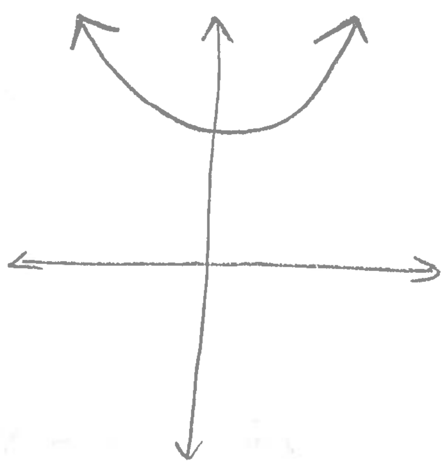
$b^2 - 4ac = (-3)^2 - 4(7)(4)$

$= 9 - 112 = -103 < 0$

$\Rightarrow$ no roots

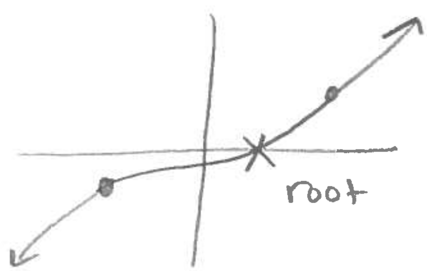
$7\left(x^2 - \frac{3}{7}x + \frac{4}{7}\right)$

not monic, factor out 7



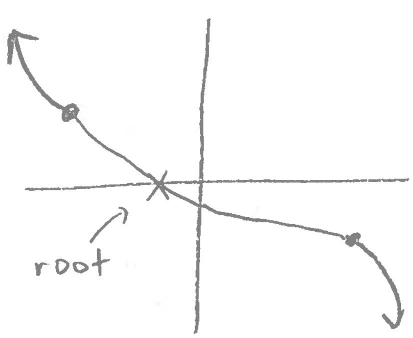
Leading Order Term $= 7x^2$

11) 2 cases: ① Leading order coefficient is positive



to connect the left and right we need to cross the x-axis where we do is a root

② Leading order coefficient is negative



same as case one, we need to cross somewhere

$\Rightarrow$ every odd degree polynomial has at least one root

Additional Problems:

1) $p(x) = x^4 + 5x^3 + 5x^2 - 5x - 6$ has roots $x = 1, -2$

$\Rightarrow$ we have factors $(x+2), (x-1)$

Note: if $(x+2)$ and $(x-1)$ are factors, then

$(x+2)(x-1) = x^2 + x - 2$ is a factor

$$
\begin{array}{r l}
 & x^2 + 4x + 3 \\
x^2 + x - 2 \,\big) & x^4 + 5x^3 + 5x^2 - 5x - 6 \\
 & -(x^4 + x^3 - 2x^2) \\
 & \quad 4x^3 + 7x^2 - 5x \\
 & \quad -(4x^3 + 4x^2 - 8x) \\
 & \qquad 3x^2 + 3x - 6 \\
 & \qquad -(3x^2 + 3x - 6) \\
 & \qquad\qquad 0
\end{array}
$$

Thus $p(x) = (x+2)(x-1)(x^2 + 4x + 3)$

check discriminant: $b^2 - 4ac$

$(4)^2 - 4(1)(3) = 4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{4}}{2}$$

$p(x) = (x+2)(x-1)(x+1)(x+3)$

$\boxed{x = 3, 1}$ are the remaining root