Counting I

For this section you'll need to know what *factorials* are.

If $n \in \mathbb{N}$, then n-factorial, which is written as $n!$, is the product of numbers

$$n(n-1)(n-2)(n-3)\cdots(4)(3)(2)(1)$$

Examples. $3! = (3)(2)(1) = 6$, and $5! = (5)(4)(3)(2)(1) = 120$.

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Options multiply

When you have to make one choice, and then another choice, the total number of choices multiply.

Suppose you have to choose a sandwich with one of four types of meat – ham, turkey, pastrami, or roast beef – and one of three kinds of cheese – swiss, cheddar, or gouda.

There are 4 different ways to choose a meat. Once you've made that choice, there are 3 different ways to choose a cheese. The number of choices for meat and cheese can be displayed in a 4×3 rectangle, where it's easy to see that the total number of choices for sandwiches is $4(3) = 12$. (12 is the area of a 4×3 rectangle.)

	Swiss	Cheddar	Gouda
Ham	H/S	H/C	H/G
Turkey	T/S	T/C	T/G
Pastrami	P/S	P/C	P/G
Roast Beef	R/S	R/C	R/G

If, in addition to a meat and cheese option, you are given a bread option of either wheat or white, then that's a third choice to make. The third choice has 2 options, and the number of options multiply, so there would be $4(3)(2) = 24$ total number of sandwiches to choose from. (You could build a rectangular solid of dimensions $4 \times 3 \times 2$, to list out the total number of options, just as we made a rectangle above to list the number of options after making two different choices. The area of a $4 \times 3 \times 2$ rectangular solid is 24.)

Examples.

• You have to wear a tie and jacket for a fancy dinner. You own 3 jackets and 7 ties. There are $3(7) = 21$ possible jacket and tie combinations to choose from.

• You are buying an airplane ticket for a flight with a meal service. When you buy your ticket, you can choose to sit in an aisle seat, or a window seat. You can choose the vegetarian meal, or the chicken. You can sit in any row of the plane you like: 1-32. How many different tickets can you buy?

There are 2 options for type of seat, 2 options for the meal, and 32 options for the row. Thus, there are $2(2)(32) = 128$ total number of different tickets you can buy.

• Suppose you are designing a house for yourself to live in. You can choose the house to be made out of wood, brick, or metal. The roof can be wood shingles, asphalt shingles, or tin. You can paint the house brown, red, yellow, or green. You can choose to have two, three, four, or five bedrooms. How many total number of possibilities are there for the design of your house?

There are 3 building material options, 3 roof options, 4 color options, and 4 bedroom options. Altogether, there are $3(3)(4)(4) = 144$ total options for the design of your house.

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Ordering sets

Remember that the order in which we list the contents of a set doesn't change what the set is. For example, $\{5, 2, 3\} = \{2, 3, 5\}$.

But sometimes it can be useful to *order* the contents of a set: that is, to designate an object of the set as being "first", and another object as being "second", etc.

Examples.

     • There are two different ways to order the objects of the set $\{\pi, b\}$. We could either choose to make π the first object, and b the second, or we could take b to be first and π second.

     • Here's a list of all the ways to order the set $\{\sqrt{2}, -\frac{2}{5}, f, e\}$. If you count the items on the list, you'll see that there are 24 different ways to order the set $\{\sqrt{2}, -\frac{2}{5}, f, e\}$.

$\sqrt{2},$	$-\frac{2}{5},$	$f,$	e	$\sqrt{2},$	$f,$	$-\frac{2}{5},$	e	$\sqrt{2},$	$e,$	$f,$	$-\frac{2}{5}$
$\sqrt{2},$	$-\frac{2}{5},$	$e,$	f	$\sqrt{2},$	$f,$	$e,$	$-\frac{2}{5}$	$\sqrt{2},$	$e,$	$-\frac{2}{5},$	f
$-\frac{2}{5},$	$\sqrt{2},$	$f,$	e	$-\frac{2}{5},$	$f,$	$\sqrt{2},$	e	$-\frac{2}{5},$	$e,$	$f,$	$\sqrt{2}$
$-\frac{2}{5},$	$\sqrt{2},$	$e,$	f	$-\frac{2}{5},$	$f,$	$e,$	$\sqrt{2}$	$-\frac{2}{5},$	$e,$	$\sqrt{2},$	f
$e,$	$\sqrt{2},$	$f,$	$-\frac{2}{5}$	$e,$	$f,$	$\sqrt{2},$	$-\frac{2}{5}$	$e,$	$-\frac{2}{5},$	$f,$	$\sqrt{2}$
$e,$	$\sqrt{2},$	$-\frac{2}{5},$	f	$e,$	$f,$	$-\frac{2}{5},$	$\sqrt{2}$	$e,$	$-\frac{2}{5},$	$\sqrt{2},$	f
$f,$	$e,$	$\sqrt{2},$	$-\frac{2}{5}$	$f,$	$\sqrt{2},$	$e,$	$-\frac{2}{5}$	$f,$	$-\frac{2}{5},$	$\sqrt{2},$	e
$f,$	$e,$	$-\frac{2}{5},$	$\sqrt{2}$	$f,$	$\sqrt{2},$	$-\frac{2}{5},$	e	$f,$	$-\frac{2}{5},$	$e,$	$\sqrt{2}$

Spelling. Arranging the order of a set of letters is a good example of when order is important. The set $\{\mathbf{e}, \mathbf{t}, \mathbf{a}\}$ can be ordered in six different ways: **eta**, **eat**, **tea**, **tae**, **ate**, and **aet**. Some of the six arrangements are not words. Some of them are words, and the words that do appear have different meanings. So if you have a set of letters, the order in which you write them is very important.

Another look at spelling. Let's find a better way to count the number of ways we can order the objects of the set $\{\mathbf{e}, \mathbf{t}, \mathbf{a}\}$ without having to write out a list of them, and then counting the list.

To arrange the three letters **e**, **t**, and **a** into some order, we need to choose a letter to be first. There are 3 letters, and thus 3 options for which letter can be first.

Once we've decided which letter is first, there are two letters remaining. We choose one of the two to be second, so there are 2 options for which number is second.

After we've chosen a first letter and a second letter, only one letter remains. It must be third, because there are no other letters to choose from. So there is only 1 option for which letter can be third at this point.

Options multiply. There were 3 options for the first letter, followed by 2 options for the second letter, and 1 option for the third letter. So the total number of ways the letters **e**, **t**, and **a** can be arranged is $3! = (3)(2)(1) = 6$.

General Problem. Suppose you have a set that contains exactly n objects. How many different ways are there to order the objects in the set?

General Solution. We have to choose a first object. There are n total objects in the set, so there are n different options for what that first object could be.

Once we've chosen a first object, we remove it from the set, leaving $(n-1)$ options for what the second object could be.

Once we've chosen and removed the first and second objects from the set, there are $(n-2)$ objects from which we could choose a third, so there are $(n-2)$ options for what object we can make third.

After we've selected and removed the first three objects, there are $(n-3)$ options left for what could be fourth.

This pattern continues. Eventually there will be two objects left for us to choose from in deciding which object will be next-to-last. That means we have 2 options at this point for what the next-to-last object will be.

After having chosen and selected what the first $(n-1)$ objects are, there is only one object from the set remaining. That means there is only 1 option for what we can take last.

We just made n different choices: a choice for first, second, third, fourth..., next-to-last, and last. Options multiply, so the total number of ways we can order a set of n objects is

$$n(n-1)(n-2)(n-3)(n-4)\cdots(2)(1) = n!$$

Examples.

• There are exactly 4 objects in the set $\{\sqrt{2}, -\frac{2}{5}, f, e\}$. Therefore, there are $4! = 24$ ways to order the objects in the set $\{\sqrt{2}, -\frac{2}{5}, f, e\}$. (Wasn't that much easier than making a list of all the options, and then counting the items on the list?)

• You have 8 rooms in your house, and 8 different lamps. You want to put a single lamp in each of the rooms, but you're not sure which one to put in which room. So you decide to try out all possible arrangement of lamps in rooms to see which arrangement you like the best.

If it takes you two minutes every time you try out a new arrangement of lamps in rooms, and if you never take a break for sleeping, eating, using the restroom, etc., then you will have finished experimenting with all of the possible arrangements of lamps after 56 days.

There are $8! = 40320$ arrangements, and each arrangement cost you 2 minutes. So the task will require $40320(2) = 80640$ minutes. There are 1440 minutes in a day, so the task will take $\frac{80640}{1440} = 56$ days.

Of course, the amount of time spent arranging lamps increases as the number of possible arrangements increase. If you were unfortunate enough to live in a 20 room mansion, and you wanted to experiment by placing one of 20 different lamps into each room in every possible arrangement, and if it took you two minutes each time you rearranged the lamps, then trying out every possible arrangement would take more than 9 trillion years – if you never stopped for a break of any kind.

• There are 26 letters in the English alphabet. We have assigned those letters an order: A,B,C,D,...,X,Y,Z, which is called the alphabetical order. This is only one of the choices for ordering the alphabet that we could have made as a society. We could have chosen any one of the possible $26!$ different ways to order the alphabet. Note that

$$26! = 403,291,461,126,605,635,584,000,000$$

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Choosing and ordering some of the objects in a set

In the Olympics you might have 120 people competing in the same event. The goal of the competition is to determine a gold, silver, and bronze athlete, and that's it. The Olympics will choose and order 3 athletes out of the 120.

We want a general formula that will allow us to count the number of different ways that we can choose and order k objects out of a set of n objects. (Of course, for this process to make sense we need to have that $k \leq n$.)

We could choose any of the n objects to be first in our order, leaving us with $(n-1)$ options for a second, then $(n-2)$ options for the third, and so on, until we have chosen the first $k-1$ objects. The last step would be to choose a kth object from the remaining $(n-(k-1)) = (n-k+1)$ objects. Then we multiply the number of options we had for each choice to find that the number of different ways that we can choose and order k objects out of a set of n objects is

$$n(n-1)(n-2)\cdots(n-k+2)(n-k+1)$$

That number is the same as the fraction

$$\frac{n(n-1)(n-2)\cdots(n-k+2)(n-k+1)(n-k)(n-k-1)\cdots(2)(1)}{(n-k)(n-k-1)\cdots(2)(1)}$$

And the above fraction can be written more simply as

$$\frac{n!}{(n-k)!}$$

Examples.

• If 120 athletes are competing for a gold, silver, and bronze medal (and no athlete can win two medals), then there are

$$\frac{120!}{(120-3)!} = \frac{120!}{117!} = \frac{120(119)(118)(117!)}{117!} = 120(119)(118) = 1,685,040$$

different ways that the athletes could be standing on the winner's podium by the end of the competition.

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Exercises

For #1-2, determine how many choices you have to make, and how many options you have for each choice. Then multiply the number of options.

1.) How many ways are there to choose a five-digit PIN code using the numbers 0-9 if no two consecutive numbers in the code are allowed to be the same? (For example, 34556 is not allowed, but 54568 is allowed.)

2.) You have three pairs of shoes, four pairs of pants, six shirts, two jackets, and two hats to choose from. How many different outfits can you put together that use one pair of shoes, one pair of pants, one shirt, one jacket, and one hat?

For #3-4, if you are ordering a set of n objects, there are $n!$ possibilities.

3.) How many ways can the letters   **a**, **f**, **t**, **e**, **r**   be arranged?

4.) A couple plans to have five children. They have decided the names of their children in advance: Sam, Sue, Terry, Robin, and Tonie. All they have left is to decide which of their children will receive which name. How many different options are there for which child is given which name?

For #5-8, if you are choosing and ordering k objects out of a set of n objects, then there are $\frac{n!}{(n-k)!}$ possibilities.

5.) A basketball team has 12 players. There are five different positions on a basketball team. A starting lineup consists of five players, each assigned to one of the five positions. How many different ways can a coach select a starting lineup?

6.) There are 51 contestants for a film contest, one for every state and the District of Columbia. The judges need to select three different contestants, one contestant as the winner, one as the runner up, and one as the winner for best newcomer. How many different ways can the judges distribute the awards?

7.) There are 10 people on a boat. One person needs to be the captain, one needs to be the first mate, and you need a person to swab the decks (no person can do more than one job). How many different ways can those three jobs be filled by the 10 people on board.

8.) A national magazine wants to rank the three most desirable states to live in – first, second, and third – and the three most undesirable states to live in – 50th, 49th, and 48th. How many rankings are possible?

9.) Is the sequence $-3, 4, 11, 18, 25, ...$ arithmetic or geometric?

10.) What is the $45th$ term of the sequence given in #9?

11.) What is the sum of the first 45 terms of he sequence given in #9?