Sums & Series

Suppose $a_1, a_2, ...$ is a sequence.

Sometimes we'll want to sum the first k numbers (also known as *terms*) that appear in a sequence. A shorter way to write $a_1 + a_2 + a_3 + \cdots + a_k$ is as

$$\sum_{i=1}^{k} a_i$$

There are four rules that are important to know when using $\sum$. They are listed below. In all of the rules, $a_1, a_2, a_3, ...$ and $b_1, b_2, b_3, ...$ are sequences and $c \in \mathbb{R}$.

Rule 1. $c\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} ca_i$

Rule #1 is the distributive law. It's another way of writing the equation

$$c(a_1 + a_2 + \cdots + a_k) = ca_1 + ca_2 + \cdots + ca_k$$

Rule 2. $\sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i = \sum_{i=1}^{k}(a_i + b_i)$

This rule is essentially another form of the commutative law for addition. It's another way of writing that

$$(a_1 + a_2 + \cdots + a_k) + (b_1 + b_2 + \cdots + b_k) = (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_k + b_k)$$

Rule 3. $\sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i = \sum_{i=1}^{k}(a_i - b_i)$

Rule #3 is a combination of the first two rules. To see that, remember that $-b_i = (-1)b_i$, so we can use Rule #1 (with $c = -1$) followed by Rule #2 to derive Rule #3, as is shown below:

$$\begin{aligned}\sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i &= \sum_{i=1}^{k} a_i + \sum_{i=1}^{k} -b_i \\ &= \sum_{i=1}^{k} (a_i + (-b_i)) \\ &= \sum_{i=1}^{k} (a_i - b_i)\end{aligned}$$

Rule 4. $\displaystyle\sum_{i=1}^{k} c = kc$

The fourth rule can be a little tricky. The number c does not depend on i — it's a constant — so $\sum_{i=1}^{k} c$ is taken to mean that you should add the first k terms in the sequence $c, c, c, c, ...$ That is to say that

$$\sum_{i=1}^{k} c = c + c + \cdots + c = kc$$

Examples.

- $\sum_{i=1}^{5} 2$ means that you should add the first 5 terms of the constant sequence $2, 2, 2, 2, 2, \ldots$. That is,

$$\sum_{i=1}^{5} 2 = 2 + 2 + 2 + 2 + 2 = 5(2) = 10$$

- $\sum_{i=1}^{20} 3 = 20(3) = 60$

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Sum of first k terms in an arithmetic sequence

If $a_1, a_2, a_3, ...$ is an arithmetic sequence, then $a_{n+1} = a_n + d$ for some $d \in \mathbb{R}$. We want to show that

$$\sum_{i=1}^{k} a_i = \frac{k}{2}(a_1 + a_n)$$

To show this, let's write the sum in question in two different ways: front-to-back, and back-to-front. That is,

$$\sum_{i=1}^{k} a_i = a_1 \quad + (a_1 + d) + (a_1 + 2d) + \cdots + (a_k - 2d) + (a_k - d) + a_k$$

and

$$\sum_{i=1}^{k} a_i = a_k \quad + (a_k - d) + (a_k - 2d) + \cdots + (a_1 + 2d) + (a_1 + d) + a_1$$

Add the two equations above "top-to-bottom" to get

$$2\sum_{i=1}^{k} a_i = [a_1 + a_k] + [a_1 + a_k] + [a_1 + a_k] + \cdots + [a_1 + a_k] + [a_1 + a_k] + [a_1 + a_k]$$

Count and check that there are exactly k of the $[a_1 + a_k]$ terms in the line above being added. Thus,

$$2\sum_{i=1}^{k} a_i = k[a_1 + a_k]$$

which is equivalent to what we were trying to show:

$$\sum_{i=1}^{k} a_i = \frac{k}{2}(a_1 + a_k)$$

Example. What is the sum of the first 63 terms of the sequence $-1, 2, 5, 8, ...$?

The sequence above is arithmetic, because each term in the sequence is 3 plus the term before it, so $d = 3$. The first term of the sequence is -1, so

$a_1 = -1$. Our formula $a_n = a_1 + (n-1)d$ tells us that $a_{63} = -1 + (62)3 = 185$. Therefore,

$$\sum_{i=1}^{63} a_i = \frac{63}{2}(-1 + 185) = \frac{63}{2}(184) = 5,796$$

Example. The sum of the first 201 terms of the sequence $10, 17, 24, 31, ...$ equals $\frac{201}{2}(10 + 1410) = \frac{201}{2}(1420) = 142,710$.

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Geometric series

It usually doesn't make any sense at all to talk about adding infinitely many numbers. But if $a_1, a_2, a_3, ...$ is a geometric sequence where $a_{n+1} = ra_n$ and $-1 < r < 1$, then we can make sense of adding all of the terms of the sequence together. (We'll give some reason why this is in the chapter "Geometric Series", after we've looked at exponential functions.)

We will use the symbols

$$\sum_{i=1}^{\infty} a_i$$

to represent adding all of the numbers in the sequence $a_1, a_2, a_3, ...$, and we call this infinite "sum" a *series*.

For the moment, let $S = a_1 + a_2 + a_3 + a_4 + \cdots$. Remember that in a geometric sequence $a_n = r^{n-1}a_1$, so we can rewrite S as

$$S = a_1 + ra_1 + r^2a_1 + r^3a_1 + \cdots$$

Using the distributive law we can multiply both sides of the line above by r:

$$rS = \qquad ra_1 + r^2a_1 + r^3a_1 + \cdots$$

Now we can subtract rS from S. If we did, the ra_1 terms in S and rS would cancel. So would the r^2a_1 terms, the r^3a_1 terms, etc. Thus, $S - rS = a_1$. Since the distributive law tells us that $S - rS = S(1-r)$, we have $S(1-r) = a_1$, or in other words, $S = \frac{a_1}{1-r}$. We have shown that

$$\sum_{i=1}^{\infty} a_i = \frac{a_1}{1-r}$$

Examples.

- The sum of the terms in the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$ equals 2. We know the sequence is geometric, follows the rule $a_{n+1} = \frac{1}{2}a_n$, and that the first term in the sequence equals 1. Thus

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

- The sum of the terms in the sequence $5, \frac{5}{3}, \frac{5}{9}, \frac{5}{27}, ...$ equals

$$\frac{5}{1 - \frac{1}{3}} = \frac{5}{\frac{2}{3}} = \frac{15}{2}$$

Caution. If $a_1, a_2, a_3, ...$ isn't geometric, or if it is but either $r \geq 1$ or $r \leq -1$, then

$$\sum_{i=1}^{\infty} a_i$$

probably doesn't make sense.

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Exercises

$3i+2$ describes a sequence. When $i=1$, we have $3(1)+2=5$. When $i=2$, we have $3(2)+2=8$. When $i=3$, we have $3(3)+2=11$. $3i+2$ is the formula for the sequence $5, 8, 11, 14, 17, \ldots$.

The sum

$$\sum_{i=1}^{4}(3i+2)$$

is what you'd get by adding the first 4 terms of the sequence described by $3i+2$. That is,

$$\sum_{i=1}^{4}(3i+2) = 5+8+11+14 = 38$$

The next three problems involve summing terms of formulas that are described by the formulas $2i-1$, i^2-2, and i^3. Find the sums.

1.) $\displaystyle\sum_{i=1}^{5}(2i-1)$ 2.) $\displaystyle\sum_{i=1}^{4}(i^2-2)$ 3.) $\displaystyle\sum_{i=1}^{3} i^3$

Find the following sums using Rule #4 from page 26.

4.) $\displaystyle\sum_{i=1}^{50} 3$ 5.) $\displaystyle\sum_{i=1}^{100} 49$ 6.) $\displaystyle\sum_{i=1}^{78}(-2)$

Just as we used $3i+2$ at the top of the page as a formula for describing a sequence, so too i is a formula for describing a sequence. The sequence described by i is a very simple arithmetic sequence. The first term is 1, the second term is 2, the third term is 3, and so on, so that the sequence is $1, 2, 3, 4, 5, 6, \ldots$. Use the formula on page 27 to find the sums below, the sums of the first 40, 100, and 900 terms of this arithmetic sequence.

7.) $\displaystyle\sum_{i=1}^{40} i$ 8.) $\displaystyle\sum_{i=1}^{100} i$ 9.) $\displaystyle\sum_{i=1}^{900} i$

10.) What is the sum of the first 701 terms of the sequence $-5, -1, 3, 7, ...$?

11.) What is the sum of the first 53 terms of the sequence $140, 137, 134, 131, ...$?

12.) What is the sum of the first 100 terms of the sequence $4, 9, 14, 19, ...$?

13.) What is the sum of the first 80 terms of the sequence $53, 54, 55, 56, ...$?

Notice that $\frac{2}{6^i}$ is a formula for a geometric sequence. When $i = 1$, $\frac{2}{6^i} = \frac{2}{6^1} = \frac{1}{3}$. When $i = 2$, $\frac{2}{6^i} = \frac{2}{6^2} = \frac{1}{18}$. When $i = 3$, $\frac{2}{6^i} = \frac{2}{6^3} = \frac{1}{54}$. The formula $\frac{2}{6^i}$ describes the geometric sequence $\frac{1}{3}, \frac{1}{18}, \frac{1}{54}, \ldots$. It's a geometric sequence whose fist term is $\frac{1}{3}$, and whose remaining terms are each found by multiplying the preceding term by $\frac{1}{6}$. That is, this a geometric sequence where $a_1 = \frac{1}{3}$ and $r = \frac{1}{6}$. Because $\frac{1}{6}$ is between -1 and 1, we have a formula (on page 28) that tells us how to find the geometric series asked for in #14 below. Find the given geometric series in #14-16.

14.) $\displaystyle\sum_{i=1}^{\infty} \frac{2}{6^i}$ 15.) $\displaystyle\sum_{i=1}^{\infty} \frac{7}{3^i}$ 16.) $\displaystyle\sum_{i=1}^{\infty} \frac{10}{2^i}$

The problems in #17-21 are asking you to find a geometric series. They are the same type of problem as those in #14-16, they just perhaps look a little different. Find the first term of the sequence (a_1), find the number that each term of the sequence is multiplied by to get the next term of the sequence (r), and then use the same formula that you used in #14-16, as long as r is a number between -1 and 1.

17.) Sum all of the terms of the geometric sequence $20, 5, \frac{5}{4}, \frac{5}{16}, ...$

18.) Sum all of the terms of the geometric sequence $120, 90, \frac{135}{2}, \frac{405}{8}, ...$

19.) Sum all of the terms of the geometric sequence $7, \frac{14}{3}, \frac{28}{9}, \frac{56}{27}, ...$

20.) Sum all of the terms of the geometric sequence $25, 15, 9, \frac{27}{5}, ...$

21.) Sum all of the terms of the geometric sequence $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, ...$

22.) If the sum of the first 3976 terms of the sequence $a_1, a_2, a_3, ...$ equals 114, then what is the sum of the first 3976 terms of the sequence $\frac{3}{2}a_1, \frac{3}{2}a_2, \frac{3}{2}a_3, ...$?

23.) If the sum of the first 20 terms of the sequence $a_1, a_2, a_3, \ldots$ equals 7, and the sum of the first 20 terms of the sequence $b_1, b_2, b_3, \ldots$ equals -13, then what is the sum of the first 20 terms of the sequence $(a_1 + b_1), (a_2 + b_2), (a_3 + b_3), \ldots$?

24.) Suppose that you expect to pay $400 for gas for your car next year, and that each year after that you plan your yearly gas expenditures will increase by $20. How much will you spend on gas in the next 8 years?

25.) Suppose you are entertaining two different job offers. Job A has a starting salary of $20,000 and assures you of a raise of $1,000 per year. Job B offers you a starting salary of $23,000, with a yearly raise of $725. Which job will pay you more over the first ten years? How much more?

26.) An oil well currently produces 5 million gallons of oil per year, but the well is drying up, and each year it will produce 60% of what it did the year before. How much oil can be produced from the well before it is completely dry?