Logarithms

If $a > 1$ or $0 < a < 1$, then the exponential function $f : \mathbb{R} \to (0, \infty)$ defined as $f(x) = a^x$ is one-to-one and onto. That means it has an inverse function.

If either $a > 1$ or $0 < a < 1$, then the inverse of the function a^x is

$$\log_a : (0, \infty) \to \mathbb{R}$$

and it's called a *logarithm* of base a.

That a^x and $\log_a(x)$ are inverse functions means that

$$a^{\log_a(x)} = x$$

and

$$\log_a(a^x) = x$$

Problem. Find x if $2^x = 15$.

Solution. The inverse of an exponential function with base 2 is $\log_2$. That means that we can erase the exponential base 2 from the left side of $2^x = 15$ as long as we apply $\log_2$ to the right side of the equation. That would leave us with $x = \log_2(15)$.

The final answer is $x = \log_2(15)$. You stop there. $\log_2(15)$ is a number. It is a perfectly good number, just like 5, -7, or $\sqrt[2]{15}$ are. With some more experience, you will become comfortable with the fact that $\log_2(15)$ cannot be simplified anymore than it already is, just like $\sqrt[2]{15}$ cannot be simplified anymore than it already is. But they are both perfectly good numbers.

Problem. Solve for x where $\log_4(x) = 3$.

Solution. We can erase $\log_4$ from the left side of the equation by applying its inverse, exponential base 4, to the right side of the equation. That would give us $x = 4^3$. Now 4^3 can be simplified; it's 64. So the final answer is $x = 64$.

Problem. Write $\log_3(81)$ as an integer in standard form.

Solution. The trick to solving a problem like this is to rewrite the number being put into the logarithm — in this problem, 81 — as an exponential whose base is the same as the base of the logarithm — in this problem, the base is 3.

Being able to write 81 as an exponential in base 3 will either come from your comfort with exponentials, or from guess-and-check methods. Whether it's immediately obvious to you or not, you can check that $81 = 3^4$. (Notice that 3^4 is an exponential of base 3.) Therefore, $\log_3(81) = \log_3(3^4)$.

Now we use that exponential base 3 and logarithm base 3 are inverse functions to see that $\log_3(3^4) = 4$.

To summarize this process in one line,

$$\log_3(81) = \log_3(3^4) = 4$$

Problem. Write $\log_4(16)$ as an integer in standard form.

Solution. This is a logarithm of base 4, so we write 16 as an exponential of base 4: $16 = 4^2$. Then,

$$\log_4(16) = \log_4(4^2) = 2$$

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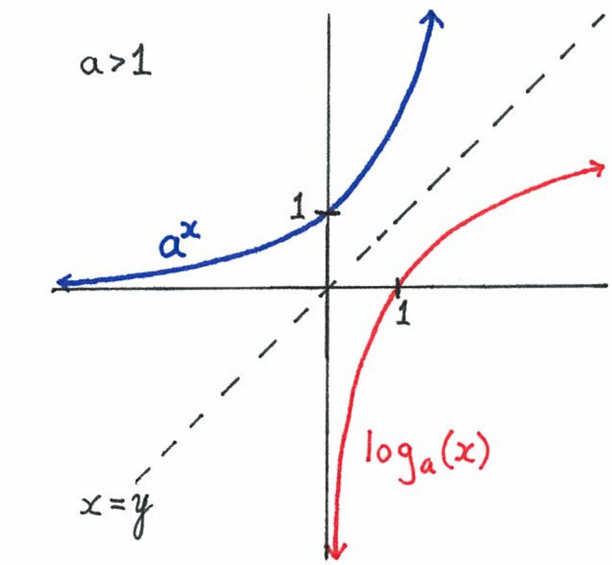
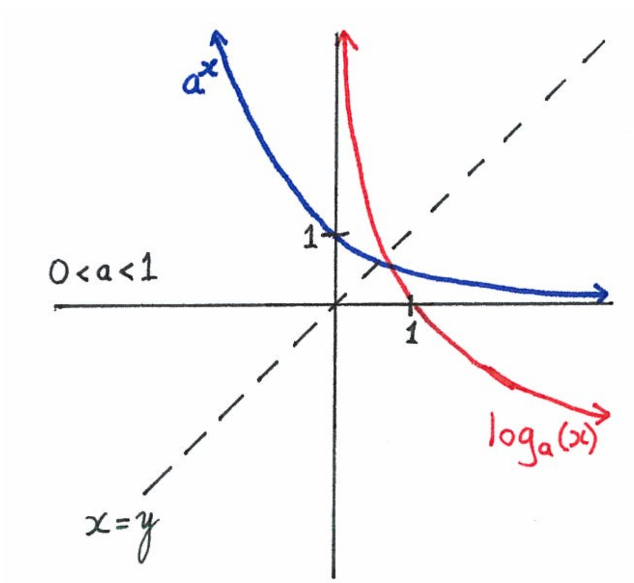
Graphing logarithms

Recall that if you know the graph of a function, you can find the graph of its inverse function by flipping the graph over the line $x = y$.

Below is the graph of a logarithm of base $a > 1$. Notice that the graph grows taller, but very slowly, as it moves to the right.



Below is the graph of a logarithm when the base is between 0 and 1.



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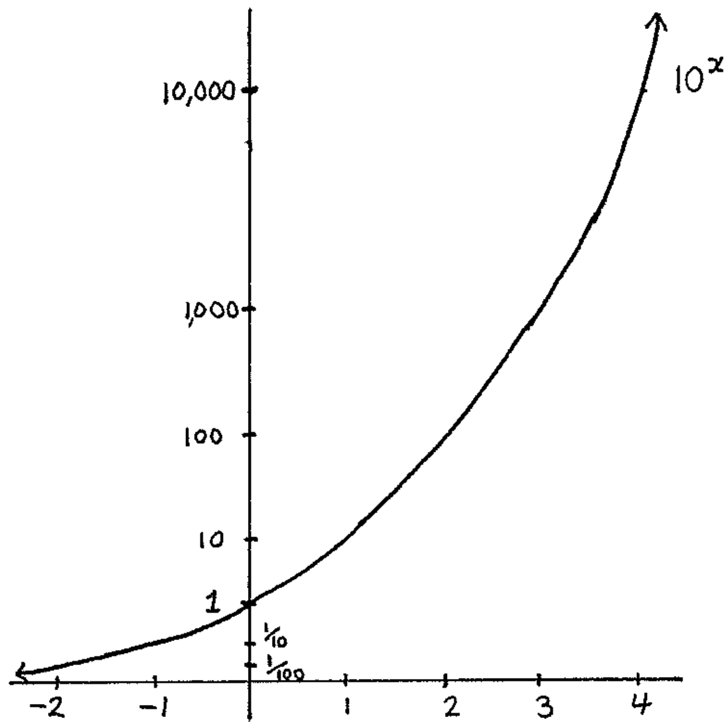
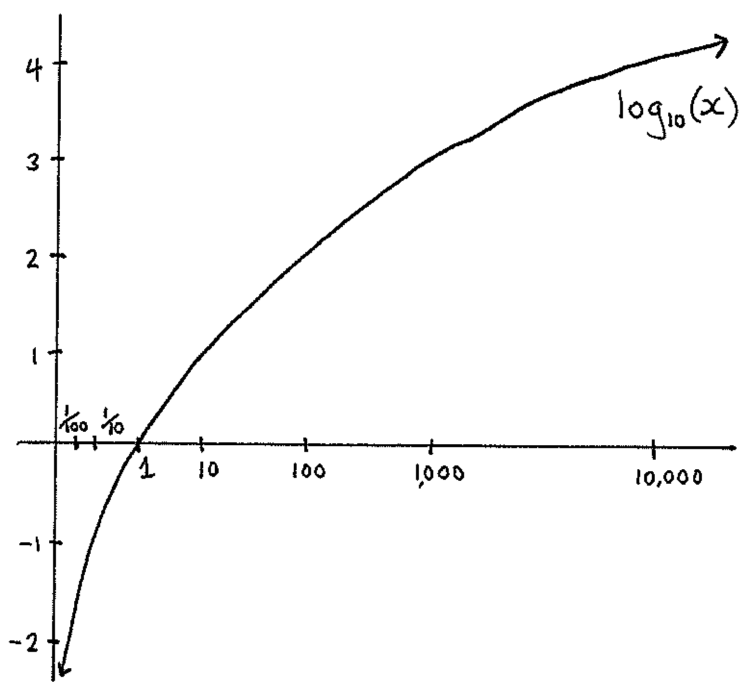
Two base examples

If $a^x = y$, then $x = \log_a(y)$. Below are some examples in base 10.

10^x	$\log_{10}(x)$
$10^{-3} = \frac{1}{1{,}000}$	$-3 = \log_{10}(\frac{1}{1{,}000})$
$10^{-2} = \frac{1}{100}$	$-2 = \log_{10}(\frac{1}{100})$
$10^{-1} = \frac{1}{10}$	$-1 = \log_{10}(\frac{1}{10})$
$10^0 = 1$	$0 = \log_{10}(1)$
$10^1 = 10$	$1 = \log_{10}(10)$
$10^2 = 100$	$2 = \log_{10}(100)$
$10^3 = 1,000$	$3 = \log_{10}(1,000)$
$10^4 = 10,000$	$4 = \log_{10}(10,000)$
$10^5 = 100,000$	$5 = \log_{10}(100,000)$

Below are the graphs of the functions 10^x and $\log_{10}(x)$. The graphs are another way to display the information from the previous chart.

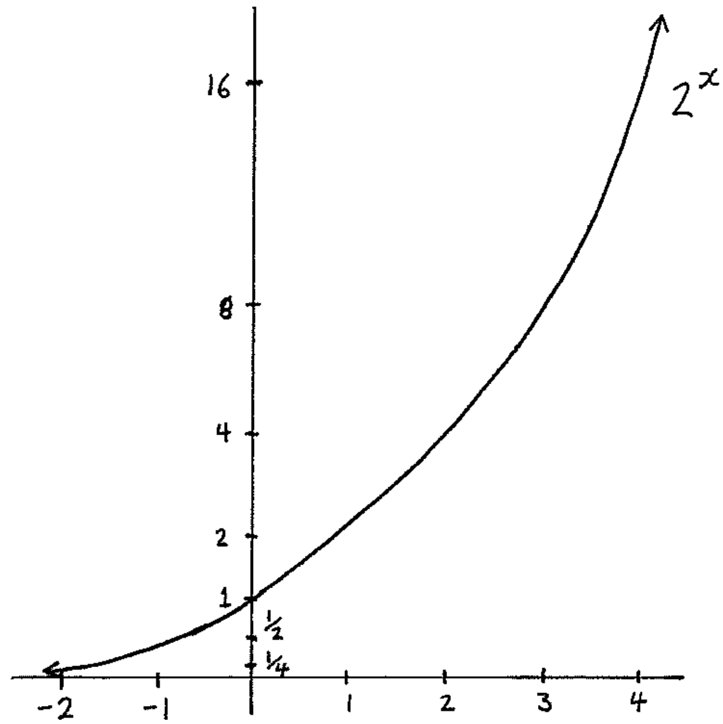


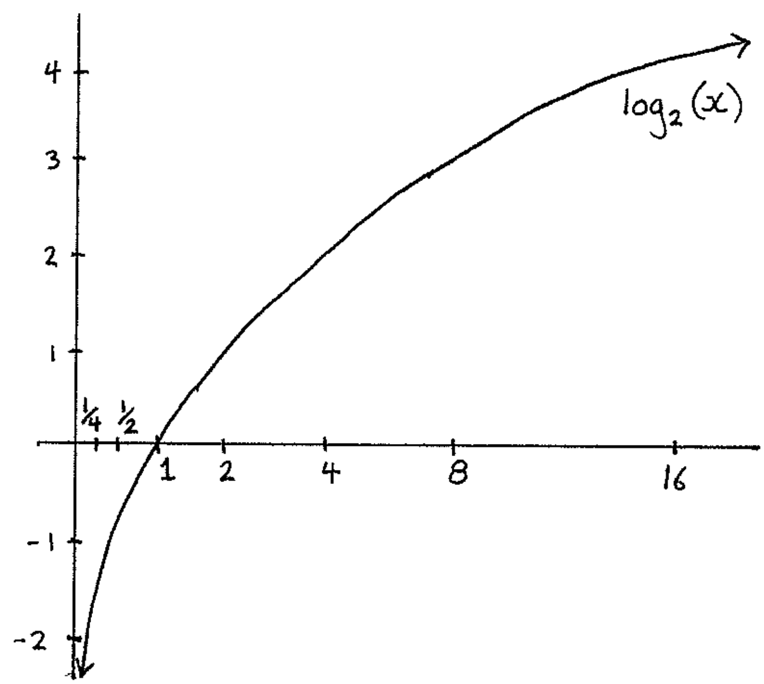


This chart contains examples of exponentials and logarithms in base 2.

2^x	$\log_2(x)$
$2^{-4} = \frac{1}{16}$	$-4 = \log_2(\frac{1}{16})$
$2^{-3} = \frac{1}{8}$	$-3 = \log_2(\frac{1}{8})$
$2^{-2} = \frac{1}{4}$	$-2 = \log_2(\frac{1}{4})$
$2^{-1} = \frac{1}{2}$	$-1 = \log_2(\frac{1}{2})$
$2^0 = 1$	$0 = \log_2(1)$
$2^1 = 2$	$1 = \log_2(2)$
$2^2 = 4$	$2 = \log_2(4)$
$2^3 = 8$	$3 = \log_2(8)$
$2^4 = 16$	$4 = \log_2(16)$
$2^5 = 32$	$5 = \log_2(32)$
$2^6 = 64$	$6 = \log_2(64)$

The information from the previous page is used to draw the graphs of 2^x and $\log_2(x)$.





Rules for logarithms

The most important rule for exponential functions is $a^x a^y = a^{x+y}$. Because $\log_a(x)$ is the inverse of a^x, it satisfies the "opposite" of this rule:

$$\log_a(z) + \log_a(w) = \log_a(zw)$$

Here's why the above equation is true:

$$\begin{aligned}\log_a(z) + \log_a(w) &= \log_a(a^{\log_a(z)+\log_a(w)}) \\ &= \log_a(a^{\log_a(z)}a^{\log_a(w)}) \\ &= \log_a(zw)\end{aligned}$$

The next two rules are different versions of the rule above:

$$\log_a(z) - \log_a(w) = \log_a\left(\frac{z}{w}\right)$$

$$\log_a(z^w) = w\log_a(z)$$

Because $a^0 = 1$, it's also true that

$$\log_a(1) = 0$$

Change of base formula

Let's say that you wanted to know a decimal number that is close to $\log_3(7)$, and you have a calculator that can only compute logarithms in base 10. Your calculator can still help you with $\log_3(7)$ because the change of base formula tells us how to use logarithms in one base to compute logarithms in another base.

The change of base formula is:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

In our example, you could use your calculator to find that 0.845 is a decimal number that is close to $\log_{10}(7)$, and that 0.477 is a decimal number that is close to $\log_{10}(3)$. Then according to the change of base formula

$$\log_3(7) = \frac{\log_{10}(7)}{\log_{10}(3)}$$

is close to the decimal number

$$\frac{0.845}{0.477}$$

which itself is close to 1.771.

We can see why the change of base formula is true. First notice that

$$\log_a(x)\log_b(a) = \log_b(a^{\log_a(x)}) = \log_b(x)$$

The first equal sign above uses the third rule from the section on rules for logarithms. The second equal sign uses that a^x and $\log_a(x)$ are inverse functions.

Now divide the equation above by $\log_b(a)$, and we're left with the change of base formula.

Base confusion

To a mathematician, $\log(x)$ means $\log_e(x)$. Most calculators use $\log(x)$ to mean $\log_{10}(x)$. Sometimes in computer science, $\log(x)$ means $\log_2(x)$. A lot of people use $\ln(x)$ to mean $\log_e(x)$. ($\ln(x)$ is called the "natural logarithm".)

In this text, we'll never write the expression $\log(x)$ or $\ln(x)$. We'll always be explicit with our bases and write logarithms of base 10 as $\log_{10}(x)$, logarithms of base 2 as $\log_2(x)$, and logarithms of base e as $\log_e(x)$. To be safe, when doing math in the future, always ask what base a logarithm is if it's not clear to you.

*   *   *   *   *   *   *   *   *   *   *   *   *

Exercises

For #1-8, match each of the numbered functions on the left with the lettered function on the right that is its inverse.

1.) $x+7$	A.) x^7
2.) $3x$	B.) $\frac{x}{3}$
3.) $\sqrt[7]{x}$	C.) 3^x
4.) 7^x	D.) $\sqrt[3]{x}$
5.) $\frac{x}{7}$	E.) $x-7$
6.) $\log_3(x)$	F.) $x+3$
7.) $x-3$	G.) $7x$
8.) x^3	H.) $\log_7(x)$

For #9-17, write the given number as a rational number in standard form, for example, 2, -3, $\frac{3}{4}$, and $\frac{-1}{5}$ are rational numbers in standard form. These are the exact same questions, in the same order, as those from #12-20 in the chapter on Exponential Functions. They're just written in the language of logarithms instead.

9.) $\log_4(16)$	10.) $\log_2(8)$	11.) $\log_{10}(10,000)$
12.) $\log_3(9)$	13.) $\log_5(125)$	14.) $\log_{\frac{1}{2}}(16)$
15.) $\log_{\frac{1}{4}}(64)$	16.) $\log_8(\frac{1}{4})$	17.) $\log_{27}(\frac{1}{9})$

For #18-25, decide which is the greatest integer that is less than the given number. For example, if you're given the number $\log_2(9)$ then the answer would be 3. You can see that this is the answer by marking 9 on the x-axis of the graph of $\log_2(x)$ that's drawn earlier in this chapter. You can use the graph and the point you marked to see that $\log_2(9)$ is between 3 and 4, so 3 is the greatest of all the integers that are less than (or below) $\log_2(9)$.

18.) $\log_{10}(15)$

19.) $\log_{10}(950)$

20.) $\log_2(50)$

21.) $\log_2(3)$

22.) $\log_3(18)$

23.) $\log_{10}(\frac{1}{19})$

24.) $\log_2(\frac{1}{10})$

25.) $\log_3(\frac{1}{10})$

For #26-33, use that $\log_a(x)$ and a^x are inverse functions to solve for x.

26.) $\log_4(x) = -2$

27.) $\log_6(x) = 2$

28.) $\log_3(x) = -3$

29.) $\log_{\frac{1}{10}}(x) = -5$

30.) $e^x = 17$

31.) $e^x = 53$

32.) $\log_e(x) = 5$

33.) $\log_e(x) = -\frac{1}{3}$

For #34-42, match the numbered functions with their lettered graphs.

34.) 10^x

35.) $(\frac{1}{2})^x$

36.) $\log_{10}(x)$

37.) $\log_{\frac{1}{2}}(x)$

38.) $\log_{10}(-x)$

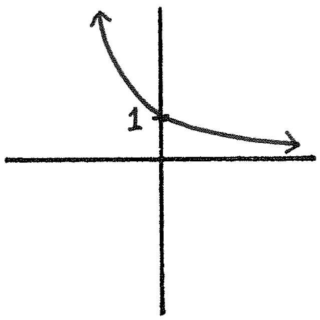
39.) $\log_{\frac{1}{2}}(x+2)$

40.) $\log_{10}(x)+2$

41.) $\log_{\frac{1}{2}}(\frac{x}{3})$

42.) $\log_{10}(x-1)$

A.)



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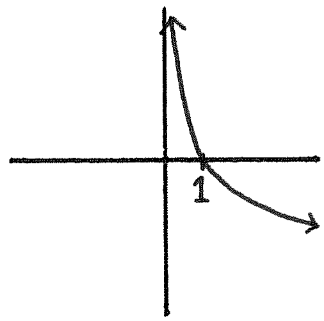

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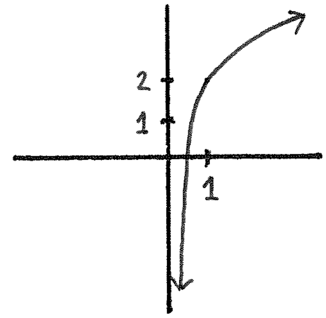

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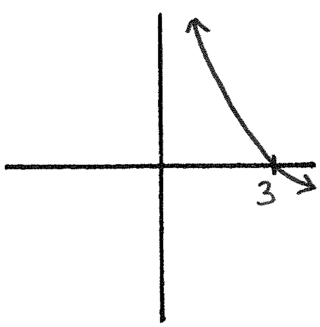

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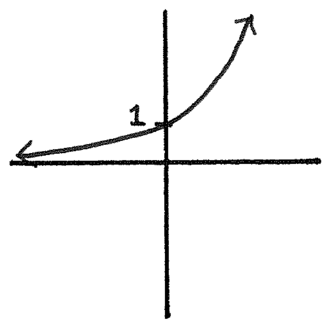

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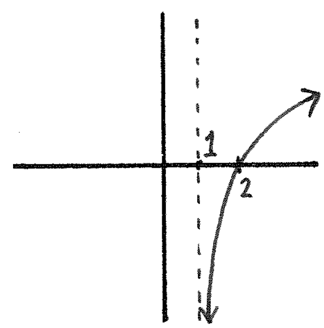
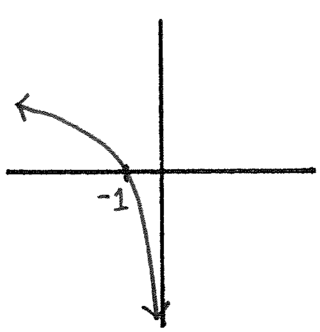
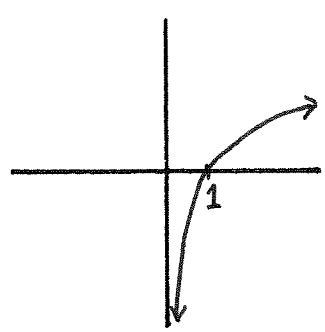


G.)



H.)



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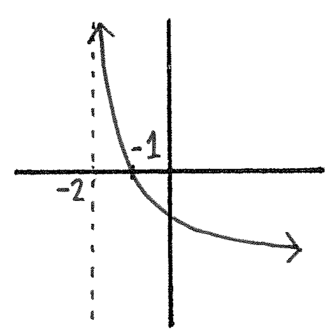
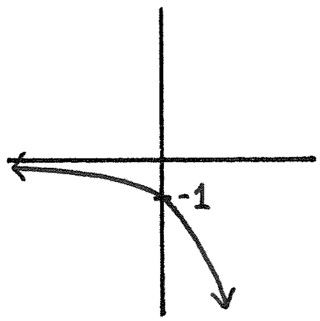


For #43-51, match the numbered functions with their lettered graphs.

43.) e^x	44.) $\log_e(x)$	45.) $\log_e(x-2)$
46.) $\log_e(x)+1$	47.) $-\log_e(x)$	48.) $-e^x$
49.) $\log_e(-x)$	50.) $\log_e(x+1)+2$	51.) $-\log_e(x+1)$

A.)



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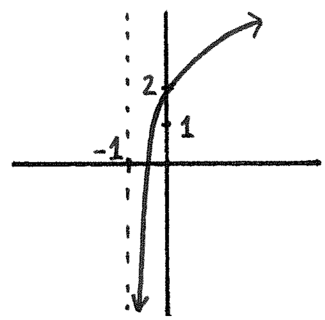

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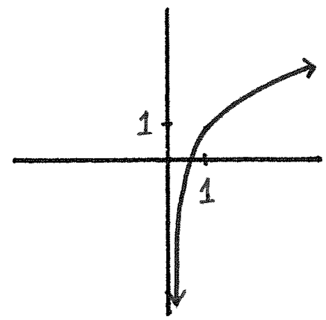

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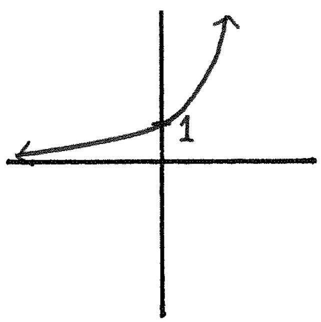

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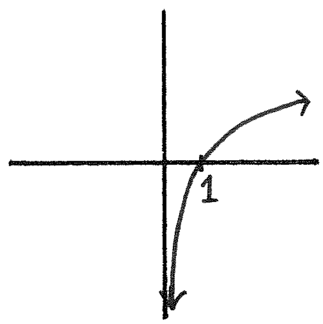

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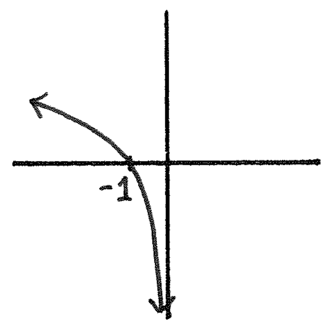

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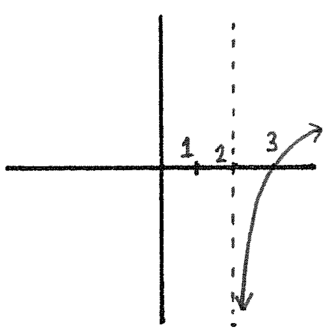

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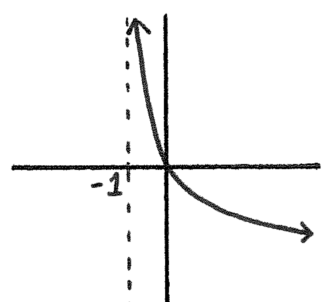

I.)