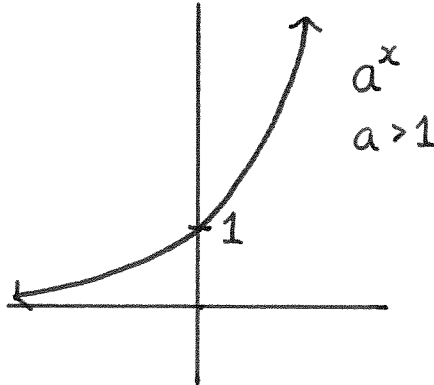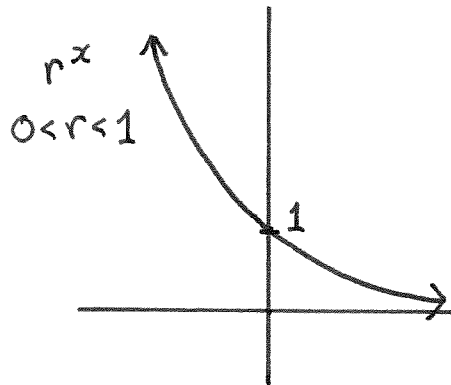# Geometric Series

In the previous chapter we saw that if $a > 1$, then the exponential function with base a, the function $f(x) = a^x$, has a graph that looks like this:

On the other hand, if $0 < r < 1$, then the exponential function of base r, the function $g(x) = r^x$, has a graph that looks like this:

Notice that larger inputs for the function r^x result in smaller outputs. Geometrically, this is represented in the graph of r^x getting closer and closer to the x-axis as we move to the right in the graph.

For example, the number $\frac{1}{2}$ is between 0 and 1. If we look at the exponential function of base $\frac{1}{2}$, then as the number x becomes larger, the number $\left(\frac{1}{2}\right)^x$ becomes closer and closer to 0. Notice that $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$, $\left(\frac{1}{2}\right)^6 = \frac{1}{64}$, $\left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$, and $\left(\frac{1}{2}\right)^{25} = \frac{1}{33,554,432}$. As the exponent becomes larger (2, 6, 10, 25), the output grows smaller, closer and closer to 0.

This discussion can be summarized by saying that if $0 < r < 1$, then as x grows larger and larger, the number r^x becomes closer and closer to 0.

In the above sentence, x is just a variable, and we can use any letter as a variable, so we'll rewrite the above sentence using the variable n rather than the variable x:

If $0 < r < 1$,
then as n grows larger and larger,
r^n becomes closer and closer to 0.

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Suppose that r is a positive number that is less than 1. That is, $0 < r < 1$. If a is a number, then a, ra, r^2a, r^3a, $r^4a, \ldots$ is a geometric sequence. Each term in the sequence is found by multiplying the previous term by the number r.

Earlier in this text we saw that if $0 < r < 1$, then the sum of all of the infinitely many terms of the geometric sequence a, ra, r^2a, r^3a, $r^4a, \ldots$ equals $\frac{a}{1-r}$. That is,

$$a + ra + r^2a + r^3a + r^4a + \cdots = \frac{a}{1-r}$$

Example. The geometric series

$$\sum_{i=1}^{\infty} \frac{4}{3^i}$$

is the infinite sum $\frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \frac{4}{81} + \cdots$ In the equation from the line just before this example, a is the first term of the sum, so here $a = \frac{4}{3}$. The number r, what each term is multiplied by to obtain the following term, is $\frac{1}{3}$. Thus, using our formula from above,

$$\sum_{i=1}^{\infty} \frac{4}{3^i} = \frac{\frac{4}{3}}{1 - \frac{1}{3}} = \frac{\frac{4}{3}}{\frac{2}{3}} = 2$$

The goal of this chapter is to use what we know about exponential functions of base r to give a better reason for why it makes sense to add the infinitely many terms of a geometric sequence if $0 < r < 1$.

$$* \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad *$$

Suppose that $0 < r < 1$ and let S_n be the sum of the first n terms of the geometric sequence a, ra, r^2a, r^3a, r^4a,.... That is,

$$S_n = a + ra + r^2a + r^3a + \cdots + r^{n-1}a$$

Using the distributive law, we see that

$$rS_n = \quad ra + r^2a + r^3a + \cdots + r^{n-1}a + r^na$$

Now we subtract rS_n from S_n. Notice that the terms ra, r^2a, r^3a, ..., $r^{n-1}a$ in each of the sums cancel, and we are left with

$$S_n - rS_n = a - r^na$$

Factoring out the S_n on the left side of the equality, and factoring out the a from the right side of the equality, we have

$$S_n(1 - r) = a(1 - r^n)$$

which we can rewrite as

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

Remember that $0 < r < 1$. As discussed previously in this chapter, if n is a really large number, then r^n is extremely close to 0. The larger n becomes, the closer r^n gets to 0. The number n in this problem is the number of terms from the sequence a, ra, r^2a, r^3a, r^4a, ... that we are summing. So as we sum more and more terms of the sequence — as n gets larger — r^n is effectively 0, and thus the sum is effectively equal to $\frac{a(1-0)}{(1-r)} = \frac{a}{1-r}$. This is what is meant by saying that we can add all of the infinitely many terms of the sequence a, ra, r^2a, r^3a, r^4a, ..., and that the result will be $\frac{a}{1-r}$.

The ideas in this chapter are good examples of concepts that are the basis of calculus. Precisely, the reasoning from the previous paragraph is called "taking a limit". In a calculus course you'll use this sort of reasoning often, and you'll be more comfortable with it every time you do.

As an aside, the discussion above could also apply to values of r with $-1 < r < 1$. We just used our assumption that $0 < r < 1$ to simplify some of our discussion here.

Exercises

1.) Suppose $0 < r < 1$. As n becomes larger and larger, which number does r^n get closer and closer to?

2.) Suppose $0 < r < 1$. As n becomes larger and larger, which number does $\frac{a(1-r^n)}{(1-r)}$ get closer and closer to?

For #3-5, determine whether the given sequences are arithmetic, geometric, or neither.

3.) $1, 1, 2, 3, 5, 8, \ldots$

4.) $13, 15, 17, 19, \ldots$

5.) $3, 6, 12, 24, \ldots$

The sequences in #6-8 are geometric sequences. For each sequence, answer the following two questions: What is the first term of the sequence? What is the number that each term of the sequence is multiplied by to find the next term in the sequence?

6.) $\frac{2}{5}, \frac{2}{25}, \frac{2}{125}, \frac{2}{625}, \ldots$

7.) $2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$

8.) $\frac{7}{3}, \frac{7}{9}, \frac{7}{27}, \frac{7}{81}, \ldots$

For #9-11, find the given geometric series. Use your answers from #6-8.

9.) $\displaystyle\sum_{i=1}^{\infty} \frac{2}{5^i}$
    10.) $\displaystyle\sum_{i=1}^{\infty} \frac{4}{2^i}$
    11.) $\displaystyle\sum_{i=1}^{\infty} \frac{7}{3^i}$

For each of the quadratic polynomials in problems #12-15:

- Complete the square.

  That means rewrite $ax^2 + bx + c$ as $a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$.

- What's the vertex of the corresponding parabola?

  The vertex of the parabola for $a(x + c)^2 + d$ is the point $(-c, d)$.

- Is its parabola opening up, or opening down?

  The parabola opens up if the leading coefficient of the quadratic polynomial is positive. The parabola opens down if the leading coefficient is negative.

- What's its discriminant?

  The discriminant of $ax^2 + bx + c$ is $b^2 - 4ac$.

- How many roots does it have?

  There are two roots if the discriminant is positive, one root if the discriminant equals 0, and zero roots if the discriminant is negative.

- What are its roots (if it has any)?

  If $ax^2 + bx + c$ has roots, they are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

- Completely factor the polynomial.

  If $ax^2 + bx + c$ has two roots, α_1 and α_2, then the completely factored form of $ax^2 + bx + c$ is $a(x - \alpha_1)(x - \alpha_2)$. If there is exactly one root, α_1, then the completely factored form is $a(x - \alpha_1)(x - \alpha_1)$. If there are no roots, then the completely factored form is $a(x^2 + \frac{b}{a}x + \frac{c}{a})$.

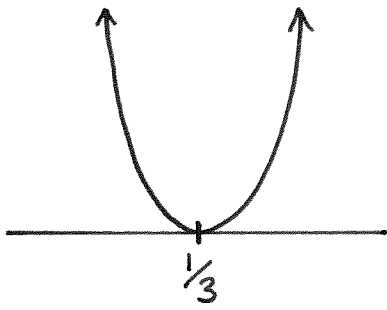- Match its graph with one of the lettered graphs on the next page.
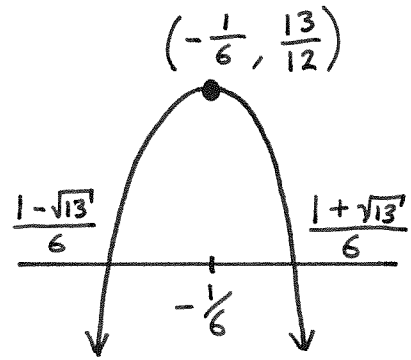

12.) $-3x^2 - x + 1$

13.) $9x^2 - 6x + 1$

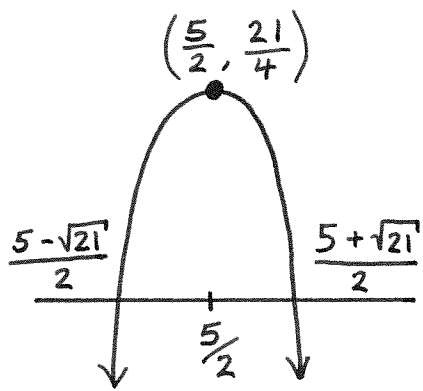14.) $2x^2 - 3x + 2$

15.) $-x^2 + 5x - 1$

A.)

$\frac{1}{3}$

B.)

$\left(-\frac{1}{6}, \frac{13}{12}\right)$

$\frac{1-\sqrt{13}}{6}$ $\frac{1+\sqrt{13}}{6}$

$-\frac{1}{6}$

C.)

$\left(\frac{5}{2}, \frac{21}{4}\right)$

$\frac{5-\sqrt{21}}{2}$ $\frac{5+\sqrt{21}}{2}$

$\frac{5}{2}$

D.)

$\frac{7}{8}$

$\frac{3}{4}$